The singular perturbation phenomenon and the turnpike property in optimal control

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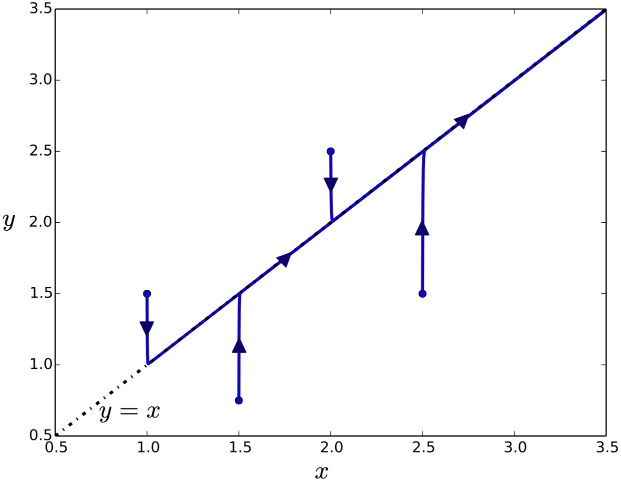
Singular perturbation: what is it ?

$$\begin{cases} \dot{x}(t) &= x(t), \qquad\qquad x(0) = x_0 \\ \varepsilon \dot{y}(t) &= x(t) - y(t), \quad y(0) = y_0 \end{cases}$$



Singularly perturbed optimal control problems

- Problem of interest:

$$(P_\varepsilon)\begin{cases} \min \int_0^1 f^0(x(t),y(t),u(t))\,dt \\ \quad \dot{x}(t) = f(x(t),y(t),u(t)), \quad x(t)\in\mathbb{R}^n, \quad x(0),\ x(1) \quad \textit{given} \\ \varepsilon\dot{y}(t) = g(x(t),y(t),u(t)), \quad y(t)\in\mathbb{R}^m, \quad y(0),\ y(1) \quad \textit{given} \end{cases}$$

where $x,\ y$ are resp. **slow** and **fast** variables since $\varepsilon > 0$ is supposed to be **small** and where $u(t)\in\mathbb{R}^k$.

- Setting $\boldsymbol{\varepsilon = 0}$, we define the *zero order reduced problem*:

$$(P_0)\begin{cases} \min \int_0^1 f^0(\overline{x}(t),\overline{y}(t),\overline{u}(t))\,dt \\ \dot{\overline{x}}(t) = f(\overline{x}(t),\overline{y}(t),\overline{u}(t)), \quad \overline{x}(0)=x(0), \quad \overline{x}(1)=x(1), \\ \qquad 0 = g(\overline{x}(t),\overline{y}(t),\overline{u}(t)). \end{cases}$$

- Roughly speaking and under suitable assumptions the main result is:

$x_\varepsilon(t)\to\overline{x}(t)$ *on* $[0,1]$ and $y_\varepsilon(t)\to\overline{y}(t)$ on every $[a,b]\subset(0,1)$, when $\varepsilon\to 0$.

Contents of the talk

- We'll first introduce the **turnpike framework** and show the link with **singularly perturbed control problems**;

- Then we'll combine the ideas developed in both approaches (turnpike property: see Trélat and Zuazua [7] and singular perturbation theory: see Khalil [4]) **and propose a path following approach to provide a more efficient numerical resolution method**;

- Finally we'll **extend our methotology to singular optimal control problems with slow variables** and give **some convergence results**.

Turnpike framework

- Let consider the optimal control problem

$$(OCP_T)\begin{cases} \min \int_0^T f^0(y(t),u(t))\,dt, & T>0 \textbf{ large enough} \\ \dot{y}(t)=f(y(t),u(t)), \quad y(t)\in\mathbb{R}^m, & u(t)\in\mathbb{R}^k, \\ y(0)=y_0, \quad y(T)=y_f. \end{cases}$$

- The associated reduced problem (or **static** optimal control problem) is

$$(SOCP_T) \qquad \min_{(y,u)\in\mathbb{R}^m\times\mathbb{R}^k} f^0(y,u) \quad \text{s.t.} \quad f(y,u)=0.$$

Turnpike property (Trélat and Zuazua [7]): under suitable assumptions, the optimal solution $(y_T(\cdot),u_T(\cdot))$ of $(OCP)_T$ remains most of the time **close to the static solution** $(\overline{y},\overline{u})$, i.e there exists positive constants C_1, C_2 such that

$$\|y_T(t)-\overline{y}\|+\|u_T(t)-\overline{u}\|\leq C_1\left(e^{-C_2 t}+e^{-C_2(T-t)}\right) \tag{1}$$

for every $t\in[0,T]$.

Example 1

$$\begin{cases} \min \frac{1}{2} \int_0^T \left[(y_1(t) - 1)^2 + (y_2(t) - 1)^2 + (u(t) - 2)^2\right] dt, & T = 20, \\ \dot{y}_1(t) = y_2(t), & (y_1(0), y_1(T)) = (1, 3) \\ \dot{y}_2(t) = 1 - y_1(t) + y_2^3(t) + u(t), & (y_2(0), y_2(T)) = (1, 0) \end{cases}$$

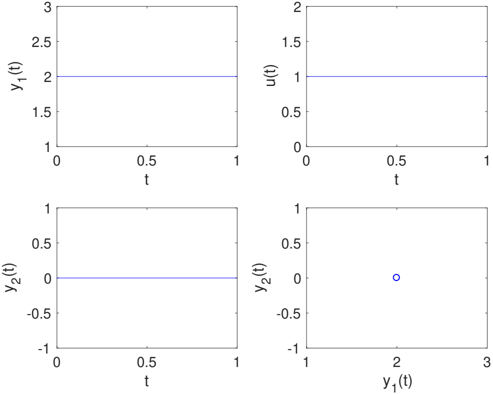


Figure: (Blue) Static solution: $(\overline{y}_1, \overline{y}_2, \overline{u}) = (2, 0, 1)$.

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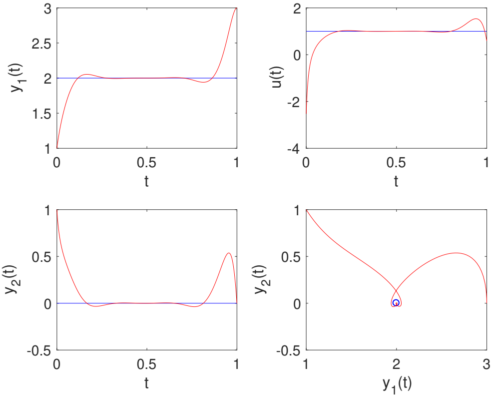

Figure: (Blue) Static solution: $(\bar{y}_1, \bar{y}_2, \bar{u}) = (2, 0, 1)$. (Red) Optimal solution compute by HamPath code.

Singular perturbation viewpoint

- Taking $s = \varepsilon t$ with $\varepsilon = 1/T$, $(OCP)_T$ becomes

$$(OCP_\varepsilon) \begin{cases} \min T \int_0^1 f^0(y(s), u(s))\, ds, \\ \varepsilon \dot{y}(s) = f(y(s), u(s)), \quad y(s) \in \mathbb{R}^m, \quad u(s) \in \mathbb{R}^k, \\ y(0) = y_0, \quad y(1) = y_f. \end{cases}$$

- Thus: Turnpike control problems $\Leftrightarrow$ singular perturbation control problems with only fast variables.
- Setting $\varepsilon = 0$, the zero order reduced system is the static problem:

$$\min_{(y,u)\in\mathbb{R}^m\times\mathbb{R}^k} f^0(y, u) \quad \text{s.t.} \quad f(y, u) = 0.$$

The **KKT conditions** of the static problem are given by the reduced (putting $\varepsilon = 0$) necessary optimality conditions of (OCP_ε) given by the Pontryagin Maximum Principle:

$$\varepsilon \dot{y} = \nabla_q H(\overline{y}, \overline{q}, \overline{u}), \quad \varepsilon \dot{q} = -\nabla_y H(\overline{y}, \overline{q}, \overline{u}), \quad 0 = \nabla_u H(\overline{y}, \overline{q}, \overline{u}),$$

where $H(y, q, u) = -f^0(y, u) + \langle q, f(y, u)\rangle$ is the pseudo-Hamiltonian.

Methodology

- **Goal:** Solve (OCP_ε) for ε **small**.

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- **Goal:** Solve (OCP_ε) for ε **small**.
- **Difficulty 1:** Choice of the **initial guess**.
- **Difficulty 2:** The singular perturbation introduces **stiffness** that makes the numerical integration difficult.
- **Methodology:**
  - Step 1: **Resolution of the KKT** conditions of the static problem;
  - Step 2: **Continuation on the boundary conditions** for sufficiently large ε ;
  - Step 3: **Continuation on** ε.

Step 2: Continuation on the boundary conditions

We define the shooting homotopic function by

$$\begin{array}{rcl} S : \mathbb{R}^m \times \mathbb{R} & \to & \mathbb{R}^m \\ (q_0, \lambda) & \mapsto & S(q_0, \lambda) = y(1, q_0) - (\lambda y_f + (1-\lambda)\overline{y}) \end{array}$$

with ε fixed and where $(y(\cdot, q_0), q(\cdot, q_0))$ is solution of

$$\begin{aligned} \varepsilon \dot{y}(t) &= f\left(y(t), u(y(t), q(t))\right) \\ \varepsilon \dot{q}(t) &= -\nabla_y H\left(y(t), q(t), u(y(t), q(t))\right) \\ y(0) &= y_0 \\ q(0) &= q_0 \end{aligned}$$

The control in feedback form $u(y, q)$ is assumed to be given by the PMP.

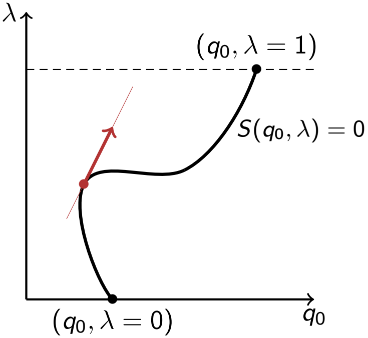

Figure: HamPath computes the path of zeros of the homotopy $S(q_0, \lambda) = 0$

Example 1 - Step 1: KKT conditions of the static problem

For $\lambda = 0$, the solution is the static solution:

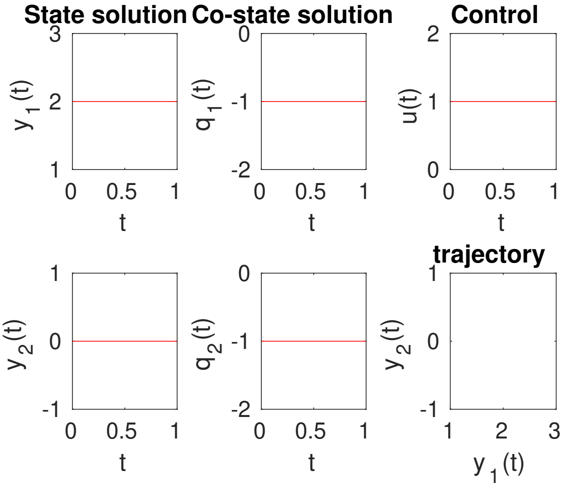

Figure: Graphs of state, co-state, control and trajectory in the plan and initial and final state taking as: $(\overline{y}_1, \overline{y}_2), (\overline{y}_1, \overline{y}_2)$

Example 1 - Step 2: Continuation on the boundary cond.

For the second step i.e making homotopy on initial and final conditions we obtain, during the evolution of λ, the different trajectories:

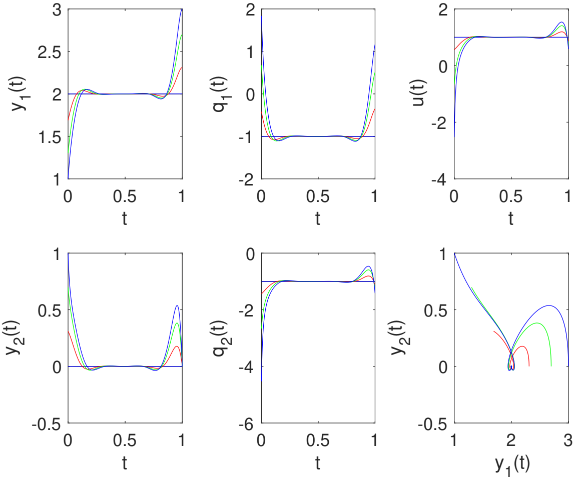

Figure: Graphs of state, co-state, control and trajectory in the plan during the homotopy on λ for **T=20** i.e $\varepsilon = 0.05$ fixed.

Example 1 - Step 3: Continuation on $\varepsilon = 1/T$

For the third step i.e making homotopy on ε and we finally obtain for $T = 70$ the following solution with a good accuracy (less than 10^{-6}):

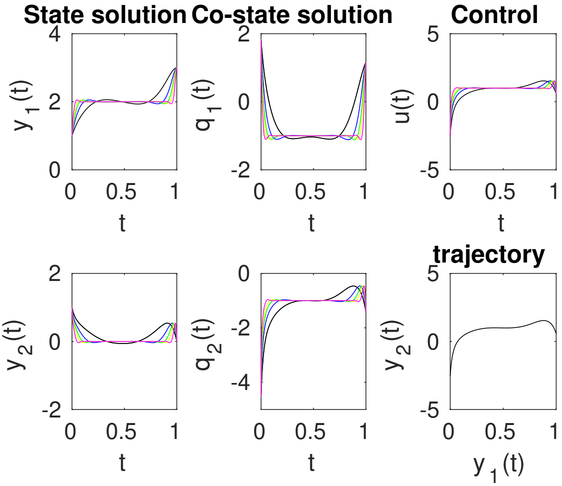

Figure: Graphs of state, co-state, control and trajectory in the plan after the homotopy on ε (with $\varepsilon_f = 1/70$), $\lambda = 1$ being fixed,

Convergence of different algorithms

Algorithms	$t_f = 20$	$t_f = 40$	$t_f = 41$	$t_f = 60$	$t_f = 70$
Simple shooting	✓ (red)	✓ (red)	✗	✗	✗
Step 2 only	✓ (green)	✓ (green)	✓ (green)	✓ (orange)	✓ (orange)
Step 2 and 3	✓ (green)	✓ (green)	✓ (green)	✓ (green)	✓ (green)

Table: We use `HamPath` for the numerical experimentations. Numerical integrations are done with the `dopri5` function with relative and absolute local errors of 1.e-8 and 1.e-14. The signification of the symbols ares :

- ✓ (green) : the algorithm converges without difficulty;
- ✓ (orange) : the algorithm converges but is very slow;
- ✓ (red) : the algorithm converges but with warnings in numerical integration;
- ✗ : the algorithm diverges.

Generalization to singularly perturbed optimal control problems

Optimal control problem with singular perturbation

- Let consider now the general problem, recalling that $u(t) \in \mathbb{R}^k$:

$$(P_\varepsilon)\begin{cases} \min \int_0^1 f^0(x(t), y(t), u(t))\,dt \\ \dot{x}(t) = f(x(t), y(t), u(t)), \quad x(t) \in \mathbb{R}^n, \quad x(0),\ x(1) \quad \textit{given} \\ \varepsilon \dot{y}(t) = g(x(t), y(t), u(t)), \quad y(t) \in \mathbb{R}^m, \quad y(0),\ y(1) \quad \textit{given} \end{cases}$$

- Setting $\varepsilon = 0$ the *zero order reduced problem* given by:

$$(P_0)\begin{cases} \min \int_0^1 f^0(\overline{x}(t), \overline{y}(t), \overline{u}(t))\,dt \\ \dot{\overline{x}}(t) = f(\overline{x}(t), \overline{y}(t), \overline{u}(t)), \quad \overline{x}(0) = x(0), \quad \overline{x}(1) = x(1), \\ \qquad 0 = g(\overline{x}(t), \overline{y}(t), \overline{u}(t)). \end{cases}$$

remains an optimal control problem, but independent of ϵ.

- **Remark:** boundary conditions are still verified on the slow variables x i.e $(\overline{x}(0), (\overline{x}(1)) = (x(0), x(1))$ but not on the fast variables y.

Boundary Value Problem

- **Remark:** Co-states $p(t)$ (resp. $q(t)$) associated to the slow variables $x(t)$ (resp. fast variables $y(t)$) are slow variables (resp. fast variables). Therefore we define slow and fast vector ψ and β given by:

$$\psi(t) = (x(t), p(t))^T, \quad \beta = (y(t), q(t))^T,$$

and we denote $[t] = (\psi(t), \beta(t))$

- Thus (BVP) which comes from the Pontryagin's Maximum Principle is

$$(BVP)_\varepsilon \begin{cases} \dot{\psi}(t) & = F[t] = \nabla_\beta H(\psi(t), \beta(t), u(\psi(t), \beta(t))), \\ \varepsilon\dot{\beta}(t) & = G[t] = -\nabla_\psi H(\psi(t), \beta(t), u(\psi(t), \beta(t))), \\ \psi_{1,\cdots,n}(0) & = x(0), \quad \psi_{1,\cdots,n}(1) = x(1), \\ \beta_{1,\cdots,m}(0) & = y(0), \quad \beta_{1,\cdots,m}(1) = y(1), \end{cases}$$

Boundary Value Problem for the first step

We denote $\overline{[t]} = (\overline{\psi}(t), \overline{\beta}(t))$

- Reduced Hamiltonian system is:

$$\begin{cases} \dot{\overline{\psi}}(t) & = F\overline{[t]}, \\ 0 & = G\overline{[t]}. \end{cases}$$

Assuming that G_β is invertible, one gets

$$\overline{\beta}(t) = \Phi(\overline{\psi}(t)),$$

more precisely:

$$\dot{\overline{\beta}}(t) = G_\beta^{-1}\overline{[t]}\left(G_t\overline{[t]} + G_\psi\overline{[t]}\ \dot{\overline{\psi}}[t]\right) \quad \text{and}$$

$$\overline{\beta}(0) = \Phi(\psi(0)), \quad \overline{\beta}(1) = \Phi(\psi(1))$$

Thus we obtain the "*zero boundary value problem*" for the first step of our algorithm:

$$\text{i.e} \quad (BVP)_0 \begin{cases} \dot{\overline{\psi}}(t) & = F\overline{[t]} \\ \dot{\overline{\beta}}(t) & = G_\beta^{-1}\overline{[t]}\left(G_t\overline{[t]} + G_\psi\overline{[t]}\,\dot{\overline{\psi}}(t)\right), \\ \overline{\psi}_i(0) & = x_i(0), \qquad \overline{\psi}_i(1) = x(1),\ i = 1, \cdots, n \\ \overline{\beta}_j(0) & = \Phi_j(\psi(0)), \quad \overline{\beta}_j(1) = \Phi_j(\psi(1)),\ j = 1, \cdots, m. \end{cases}$$

Methodology

- The main goal remains to solve $(BVP)_\varepsilon$. The algorithm is then

Step 1, ε fixed
Solve $(BVP)_0$ by a simple shooting

$\lambda \in [0,1]$ →

Step 2, ε fixed
Homotopy on the boundary value problem
$(BVP)_0 \to (BVP)_\varepsilon$

$\varepsilon_0 \to \varepsilon_f$ →

Step 3, $\lambda = 1$
homotopy on ε
large $\to$ ε small

$$\left\{\begin{array}{ll} \dot{\overline{\psi}}(t) & = F\overline{[t]} \\ \dot{\overline{\beta}}(t) & = G_\beta^{-1}\overline{[t]}\left(G_t\overline{[t]} + G_\psi\overline{[t]}\,\dot{\overline{\psi}}(t)\right), \\ \overline{\psi}_{1,\cdots,n}(0) & = x_i(0), \\ \overline{\psi}_{1,\cdots,n}(1) & = x(1), \\ \overline{\beta}_{1,\cdots,m}(0) & = \Phi_j(\psi(0)), \\ \overline{\beta}_{1,\cdots,m}(1) & = \Phi_j(\psi(1)). \end{array}\right. \longrightarrow \left\{\begin{array}{ll} \dot{\psi}(t) & = F[t], \\ \varepsilon\dot{\beta}(t) & = G[t], \\ \psi_{1,\cdots,n}(0) & = x(0), \\ \psi_{1,\cdots,n}(1) & = x(1), \\ \beta_{1,\cdots,m}(0) & = y(0), \\ \beta_{1,\cdots,m}(1) & = y(1), \end{array}\right.$$

Example 2: Step 1, $\varepsilon = 0$

$$\begin{cases} \min & J(u) = \int_0^1 \left(x^4(s) + \frac{1}{2}y^2(s) + \frac{1}{2}u^2(s)\right) ds. \\ \dot{x}(s) = x(s)y(s), & (x(0), x(1)) = \left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right) \\ \varepsilon\dot{y}(s) = -y(s) + u(s), & (y(0), y(1)) = (0, 0) \end{cases}$$

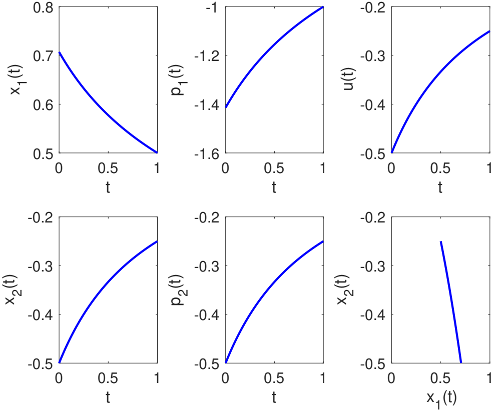

Figure: Graph of state, co-state, control and trajectory in the plan for $(BVP)_0$

Example 2: Step 2

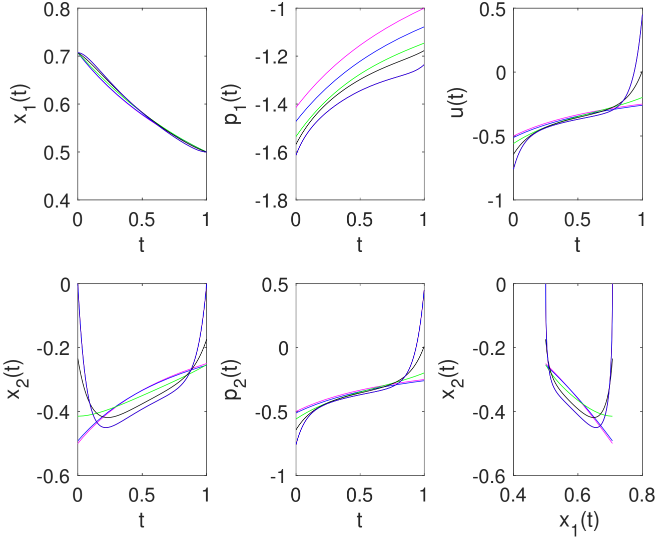

Figure: Graph of state, co-state, control and trajectory in the plan during the homotopy

Example 2: Step 3

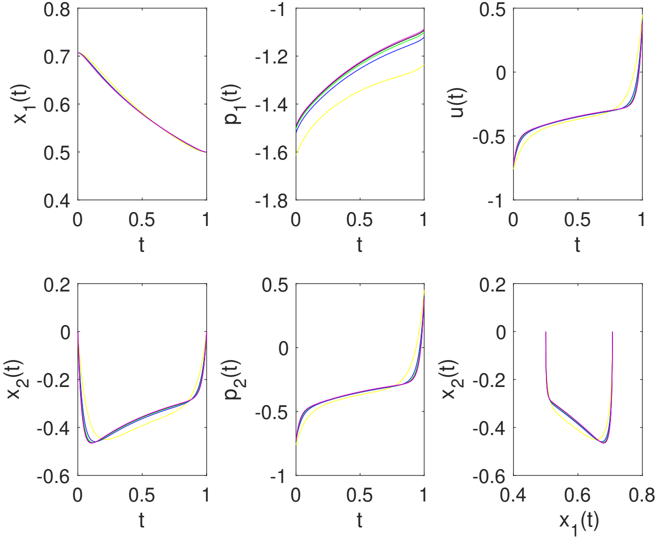

Figure: Graph of state, co-state, control and trajectory in the plan after the last homotopy; $\varepsilon = 0.04$

Convergence of different algorithms

Algorithms	$\varepsilon = 0.09$	$\varepsilon = 0.08$	$\varepsilon = 0.06$	$\varepsilon = 0.027$	$\varepsilon = 0.024$
Simple shooting	✓ (green)	✓ (red)	✗	✗	✗
Step 2 only	✓ (green)	✓ (green)	✓ (green)	✓ (green)	✗
Step 2 and 3	✓ (green)	✓ (green)	✓ (green)	✓ (green)	✓ (green)

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Summary of the results

Turnpike phenomenon:

- Static optimal control problem $(SOCP)_T$:

$$\min_{\substack{(y,u)\in\mathbb{R}^n\times\mathbb{R}^k \\ f(y,u)=0}} f^0(y,u).$$

- Static optimal point $(\overline{y},\overline{u})$ solution of $(SOCP)_T$
- Assume that $(SOCP)_T$ has one solution $(\overline{y},\overline{q},\overline{u})$, that G_β is invertible then $\exists\, C_1, C_2 > 0$ such that:

$$\|y_T(t)-\overline{y}\| + \|u_T(t)-\overline{u}\| \leq C_1\left(e^{-C_2 t} + e^{-C_2(T-t)}\right) \quad \forall\, t\in[0,T]$$

Singular perturbation:

- zero reduce order problem (P_0):

$$\begin{cases} \min \int_0^1 f^0(\overline{x}(t),\overline{y}(t),\overline{u}(t))\,dt \\ \dot{\overline{x}}(t) = f(\overline{x}(t),\overline{y}(t),\overline{u}(t)), \quad \overline{x}(0)=x(0), \\ \quad 0 = g(\overline{x}(t),\overline{y}(t),\overline{u}(t)), \quad \overline{x}(1)=x(1). \end{cases}$$

- zero outer solution $(\overline{x}(t),\overline{y}(t),\overline{u}(t))$ of $(P)_0$
- Assume that G_β is invertible and that $(P)_0$ has an unique solution, then one gets

$$\begin{aligned} x(t,\varepsilon) &\to \overline{x}, \text{ on } [0,1] \\ y(t,\varepsilon) &\to \overline{y}, \text{ on } [a,b]\subset(0,1) \end{aligned}$$

with $u(t,\varepsilon) = \overline{u} + u_i(\tau) + u_f(\sigma) + O(\varepsilon)$.

Conclusion and perspectives

- **Conclusion:**
  - Although research in both framework apparently distant settings to date, the links between them suggest that a mix of the ideas of both could lead to a more general theory for solving singularly perturbed control problems in the general non-linear case.
  - Thanks to Homotopy method to obtain the numerical solutions.
- **Perspectives:**
  - Used a stiff integrator to compute the shooting function.
  - Numerical comparisons with codes for solving stiff Boundary Value Problem: `COLNEW` from U. Ascher and al., `HAGRON` from J. R. Cash and M. H. Wright.

[1] D.Cass, Optimal growth in an aggregative model of capital accumulation, a turnpike theorem, Econometrica Vol. 34 (1966), no. 4, 833–850.

[2] J. R. Cash and M. H. Wright, A Differed correction method for nonlinear two-points boundary value problems: implementation and numerical evaluation, SIAM Journal on Scientific Computing, Vol. 12 (1991), no. 4, 971–989.

[3] J. H. Chow A Class of Singularly Perturbed Nonlinear Fixed-Endpoint Control Problems Journal of Optimization Theory and Applications: Vol. 29 (1979), No. 2, 231–251. g

[4] H. K. Khalil Non-Linear systems, Prentice-Hall: Upper Saddle River, second edn 1996.

[5] P.V. Kokotovic, H. K. Khalil and J. O'Reilly, Singular Perturbation Methods in Control, Birkhäuser, Mathematics: Theory and Applications, second edn 1988.

[6] O'Mailley, Singular Perturbation Methods for Ordinary Differential Equations, Springer-Verlag: Applied Math science, Vol. 89, 1991.

[7] E. Trélat, E. Zuazua, The turnpike property in finite-dimensional nonlinear optimal control, J. Differential Equations, Vol. 258 (2015), no. 1, 81–114.