About the minimal time control of passive tracers in presence of a single point vortex

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***Abstract*— In this work, we are interested in time-optimal displacements of passive tracers (that is point vortices with zero circulation) in presence of a single point vortex (solution of the two-dimensional incompressible Euler equation). This problem from fluid mechanics, is formulated as an optimal control problem in Mayer form that can be seen as a Zermelo-like deformation (presence of a drift) of a Riemannian situation. We first prove the controllability of the control system and the existence of optimal solutions, and then we apply the Pontryagin Maximum Principle (PMP) in order to reduce the set of candidates as minimizers. The transcription of the PMP gives us a set of nonlinear equations to solve, the so-called shooting equations. We numerically check the absence of conjugate points along the computed extremals in relation with second-order conditions of local optimality. This is a first step before computing the optimal synthesis which essentially depends on the existence of abnormal extremals and which consists, for a part, in computing the cut locus, where the extremals ceases to be optimal.**

I. INTRODUCTION

In this work we are interested in controlling the displacement of particles in interaction with point vortices, in a two-dimensional fluid, and neglecting the viscous diffusion, which is equivalent to using the Euler equation instead of the Navier-Stokes equation as the mathematical model of the fluid flow. We refer to Ref. [6] for details about vortex theory. More precisely, we are interested in controlling the displacement of particles in an optimal way (we consider the minimal time control problem) and we refer to Ref. [8] for details about flow control problems. In Ref. [8], a large review of the state of the art in the field of control of vortex dynamics is portrayed, with a particular interest on problems governed by two-dimensional incompressible Euler equations. We want to emphasize the fact that compared to Ref. [11], the problem is not of controlling the position of point vortices but controlling particles moving around point vortices. The particles we consider are more generally what we call passive tracers, that is by definition, point vortices with zero circulation.

Let us describe the mathematical modelling of vortex dynamics, see Ref. [6] for more details, that will give us the dynamical model of our problem. In the case of a two-dimensional fluid, the incompressible Euler equations are given by

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)\, v = -\nabla p, \quad \nabla \cdot v = 0, \tag{1}$$

where v stands for $v(X,t) := (v_1(X,t), v_2(X,t))$ and represents the velocity field and where p is the pressure of the fluid. Due to $\nabla \cdot v = 0$ (the incompressibility equation) from (1), one can write $v = (v_1, v_2) =: (\partial_y \Psi, -\partial_x \Psi)$ where Ψ is called the stream function. Besides, let $\mathbf{w}$ denote the viscosity vector and introduce $\tilde{v} := (v, 0)$, then $\mathbf{w}$ is given by the relation $\mathbf{w} = \nabla \wedge \tilde{v} = (0, 0, \partial_x v_2 - \partial_y v_1) =: (0, 0, \omega)$, and with the two previous formulas, one can deduce the Poisson equation satisfied by Ψ, that is

$$\nabla^2 \Psi = -\omega. \tag{2}$$

The resolution of the Poisson equation coupled with the hypothesis of a finite number N of vortices allows us to write the vortex dynamics in the form:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{k_j}{2\pi} \frac{y_i - y_j}{r_{ij}^2}, \quad \frac{\mathrm{d}y_i}{\mathrm{d}t} = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{k_j}{2\pi} \frac{x_i - x_j}{r_{ij}^2}, \tag{3}$$

where (x_i, y_i) and k_i are respectively the space coordinates and circulation of the ith-vortex, and where $r_{ij}^2 := (x_i - x_j)^2 + (y_i - y_j)^2$ is the square distance between the vortices i and j. Indeed, firstly, the solution of eq. (2) is given by

$$\Psi(X) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln(\|X - Y\|)\, \omega(Y)\, \mathrm{d}Y,$$

where $\|\cdot\|$ is the Euclidean norm, and since the viscosity is concentrated in a finite number N of vortices, then we have

$$\omega(t, X) = \sum_{i=1}^{N} k_i \delta(X - X_i(t)),$$

where δ is the Dirac mass. These two last relations lead us to the vortex dynamics written in (3).

The presence of vortices in a fluid can greatly facilitate the transport of particles, which is why vortices analysis is of considerable interest in fluid mechanics. In this sense, several studies have been carried out and we refer to Refs. [6], [8], [9], [11] and the references herein for details about the use of vortex methods in control of fluid flows, about stabilization of vortex configurations, etc. As mentionned above, the aim of this paper is not to control the vortices but the displacement of particles. The idea is therefore to consider a particle (or passive tracer) as a point vortex with zero circulation and to apply a small amplitude control acting only on the passive tracer [8]. We then define the following

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optimal control problem: minimize the transfer time to steer a passive tracer from a fixed initial position to a given final target. The aim of this paper is thus to initiate the analysis of this minimal time control problem in the case of a single point vortex with nonzero circulation, that is for $N = 2$, but with $k_1 = 0$ (the first vortex is the passive tracer). In this particular case, the (second) point vortex is static and we have a two-dimensional control system of the form:

$$\dot{x}(t) = F_0(x(t)) + u_1(t)F_1(x(t)) + u_2(t)F_2(x(t)),$$

with $x(t) := (x_1(t), x_2(t)) \in \mathbb{R}^2$ the space coordinates of the passive tracer and where F_0 is deduced from (3) (it is given in the following section). The time-optimal control problem associated to this affine control system may be seen as a Zermelo-like [12] deformation (presence of the drift F_0) of a Riemannian situation. Moreover, since the control fields F_1 and F_2 will be simply given by $F_1 := \frac{\partial}{\partial x_1}$ and $F_2 := \frac{\partial}{\partial x_2}$, then it is actually a deformation of the simple Euclidean case. However, the deformation of such a simple case may lead to complex situations and is of particular interest. We refer to Ref. [10] for more details about Zermelo-like deformations.

In this context, classical questions arise and the article is organized as follows. We first prove in the section II the controllability of the control system and the existence of time-optimal solutions. In the section III, we parameterize the extremals, solution of the Pontryagin Maximum Principle, which leads to the definition of the shooting function in the section IV. The well-posedness of the shooting equation is related to the concept of conjugate points, in relation with second-order conditions of optimality, that we recall in the same section. In the section V, we numerically compute solutions of shooting equations for some chosen examples, and check the absence of conjugate points along the associated extremals, proving the local optimality of them.

II. Controllability results and existence of time-optimal solutions

A. Controllability results

In the following, we consider a single vortex, centered in the reference frame. The controlled dynamics of the passive tracer is thus given by:

$$\dot{x}_1(t) = -\frac{k}{2\pi}\frac{x_2(t)}{r^2(t)} + u_1(t), \quad \dot{x}_2(t) = \frac{k}{2\pi}\frac{x_1(t)}{r^2(t)} + u_2(t),$$

with $r(t)^2 := x_1(t)^2 + x_2(t)^2$ the square distance of the passive tracer to the point vortex and k the circulation of the point vortex. This control system is written in the following form:

$$\dot{x}(t) = F_0(x(t)) + \sum_{i=1}^{2} u_i(t)F_i(x(t)), \tag{4}$$

where the drift is given by

$$F_0(x) := \frac{\mu}{x_1^2 + x_2^2}\left(-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}\right), \quad \mu := \frac{k}{2\pi},$$

and where the control fields are simply

$$F_1 := \frac{\partial}{\partial x_1}, \quad F_2 := \frac{\partial}{\partial x_2}.$$

Let $u_{\max} > 0$ denotes the maximal control amplitude, that is $\|u(t)\| \leq u_{\max}$. Up to a time reparameterization and a rescaling of μ, one can fix $u_{\max} = 1$. We thus consider the admissible control laws in the set

$$\mathscr{U} := \{u : [0, +\infty) \to \mathbf{U};\ u \text{ measurable}\},$$

where $\mathbf{U} := \bar{B}(0,1) \subset \mathbb{R}^2$ denotes the unit Euclidean closed ball. Note that the drift F_0 introduces a singularity at the origin: the state space is thus $M := \mathbb{R}^2 \setminus \{0\}$ when $\mu \neq 0$. Let $u \in \mathscr{U}$ and $x_0 \in M$, we denote by $x_u(\cdot, x_0)$ the unique solution of (4) associated to the control u such that $x_u(0, x_0) = x_0$.

Let us recall some basic definitions and results to assert the controllability of the control system (4). We first introduce for $T > 0$ and $x_0 \in M$, the set $\mathscr{U}_{T,x_0} \subset \mathscr{U}$ of controls $u \in \mathscr{U}$ such that the associated trajectory $x_u(\cdot, x_0)$ is well defined over $[0,T]$. Then, we denote by $\mathscr{A}(T, x_0) := \{x_u(T, x_0);\ u \in \mathscr{U}_{T,x_0}\}$ the atteignable set (or reachable set) from x_0 in time T and by $\mathscr{A}(x_0) := \cup_{T \geq 0}\mathscr{A}(T, x_0)$ the atteignable set from x_0. The control system is called *controllable from* x_0 if $\mathscr{A}(x_0) = M$. It is said *controllable* if $\mathscr{A}(x_0) = M$ for any $x_0 \in M$. As a simple consequence of Chow's theorem (see [5, Theorem 5.10]), we have

Proposition 1. *The control system* (4) *is controllable.*

Proof. The state space $M \subset \mathbb{R}^2$ is a connected submanifold. The convex control domain $\mathbf{U} = \bar{B}(0,1) \subset \mathbb{R}^2$ contains a neighborhood of the origin. The (Lie) bracket-generating condition is satisfied since $\dim \operatorname{span}\{F_1(x), F_2(x)\} = 2$ for any $x \in M$. There exists a recurrent field: the drift F_0 is a rotation. This fact is clear in polar coordinates:

$$\dot{r}(t) = v_1(t), \quad \dot{\theta}(t) = \frac{\mu}{r(t)^2} + \frac{v_2(t)}{r(t)}, \tag{5}$$

where the control v is given by $v := ue^{-i\theta}$. All the hypotheses of Chow's theorem are satisfied and the result follows. □

Remark 1. We can easily construct an admissible solution. Let us fix $(x_0, x_f) \in M^2$ and denote by $r_f := \|x_f\|$ the final distance. From x_0, apply a constant control $v(t) = (1,0)$ until the distance r_f is reached and then apply a constant control $v(t) = (0,1)$ until the target x_f is reached.

B. Existence of time-optimal solutions

Given a pair $(x_0, x_f) \in M^2$ and a parameter $\mu \in \mathbb{R}$, we focus on the problem of driving (4) in minimal time from x_0 to the target x_f:

$$(P) \qquad V(x_0, x_f, \mu) := \inf_{(T,u) \in \mathscr{D}} T \quad \text{s.t.} \quad x_u(T, x_0) = x_f,$$

where $\mathscr{D} := \{(T, u) \in [0, +\infty) \times \mathscr{U};\ u \in \mathscr{U}_{T,x_0}\}$. We emphasize the fact that the *value function* V depends on the initial condition x_0, the target x_f and the parameter μ. Since the system (4) is controllable, then there exists a pair $(T^*, u^*) \in \mathscr{D}$ such that $x_{u^*}(T^*, x_0) = x_f$ with $T^* < +\infty$ and so $V(x_0, x_f, \mu) < +\infty$. To prove the existence of time-optimal solutions, one can restrict $T \in [0, T^*]$, where T^* depends on x_0, x_f and μ. We have:

Theorem 1. *For any $(x_0, x_f, \mu) \in M^2 \times \mathbb{R}$, the problem (P) admits an optimal solution.*

The proof relies on the classical Filippov's theorem [3]. Note that when $\mu = 0$, the result is clear. When $\mu \neq 0$, the idea is to prove that the problem (P) is equivalent to the same problem with the restriction that the trajectories remain in a compact set. To do this, we need two lemmas.

For the first lemma, we introduce some notations. For a pair trajectory-control (x,u) we associate the pair (q,v), $q := (r,\theta)$, in polar coordinates, cf. the proof of proposition 1. We denote by $q_v(\cdot, q_0)$ the solution of (5) with control v such that $q_v(0,q_0) = q_0$. We define for $(\varepsilon, R, \mu) \in (\mathbb{R}_+)^2 \times \mathbb{R}^*$ and $\theta_0 \in \mathbb{R}$, two optimization problems. The minimum time to make a complete turn around the vortex:

$$T_\theta(R, \theta_0, \mu) := \inf_{(T,u) \in \mathscr{D}} T \text{ s.t. } q_v(T, (R, \theta_0)) = (R, \theta_0 + s\,2\pi),$$

where $s := \mathrm{sign}(\mu)$, and the minimum time to reach the sphere of radius ε from the sphere of radius R:

$$T_r(\varepsilon, R, \theta_0, \mu) := \inf_{(T,u) \in \mathscr{D}} T \text{ s.t. } r_v(T, (R, \theta_0)) = \varepsilon.$$

Since it is clear, due to the rotational symmetry of the problem, that $T_\theta(R, \cdot, \mu)$ and $T_r(\varepsilon, R, \cdot, \mu)$ are invariant, one can fix $\theta_0 = 0$ and set $T_\theta(R,\mu) := T_\theta(R, 0, \mu)$ and $T_r(\varepsilon, R, \mu) := T_r(\varepsilon, R, 0, \mu)$.

Lemma 1. *For any (μ, ε, R) s.t. $\mu \neq 0$, $0 < R < R_\mu := \frac{|\mu|}{2\pi - 1}$ and $0 < \varepsilon < \varepsilon_{\mu,R} := R(1 - \frac{2\pi R}{|\mu| + R})$, we have*

$$T_\theta(R,\mu) < T_r(\varepsilon, R, \mu).$$

Proof. It is clear from (5), that $T_\theta(R,\mu)$ is given by the control $v(t) = (0,s)$, $s := \mathrm{sign}(\mu)$. This gives $T_\theta(R,\mu) = \frac{2\pi R^2}{|\mu|+R}$. It is also clear that $T_r(\varepsilon, R, \mu)$ is given by $v(t) = (-1, 0)$. Thus, $T_r(\varepsilon, R, \mu) = R - \varepsilon$. Fixing $\varepsilon = 0$, we have

$$T_\theta(R,\mu) = T_r(0, R, \mu) \Leftrightarrow R = \frac{|\mu|}{2\pi - 1} =: R_\mu.$$

Besides, we have

$$T_\theta(R,\mu) < T_r(\varepsilon, R, \mu) \Leftrightarrow \varepsilon < R\left(1 - \frac{2\pi R}{|\mu| + R}\right) =: \varepsilon_{\mu,R}$$

and $0 < \varepsilon_{\mu,R} \Leftrightarrow R < R_\mu$, whence the conclusion. ☐

Definition 1. An *admissible trajectory* is a trajectory x associated to a pair $(T,u) \in \mathscr{D}$, that is, such that $x = x_u(\cdot, x_0)$, satisfying the constraints $x(T) = x_f$. Let x_1 and x_2 denote two admissible trajectories and let T_1 and T_2 denote, respectively, the first positive time such that $x_1(T_1) = x_f$, $x_2(T_2) = x_f$. Then, we say that x_1 is *strictly better* than x_2 if $T_1 < T_2$.

Let $(x_0, x_f, \mu) \in M^2 \times \mathbb{R}^*$ and set $r_0 := \|x_0\|$, $r_f := \|x_f\|$.

Lemma 2. *There exists $\varepsilon > 0$ s.t. for any admissible trajectory which has a contact with $\bar{B}(0,\varepsilon)$, one can construct a strictly better admissible trajectory contained in $M \setminus B(0,\varepsilon)$.*

Proof. Let consider (ε, R) s.t. $0 < R < \min\{R_\mu, r_0, r_f\}$ and $0 < \varepsilon < \varepsilon_{\mu,R}$. Let us recall that $\varepsilon < \varepsilon_{\mu,R} < R$ since $R < R_\mu$ and consider an admissible trajectory x having a contact with $\bar{B}(0,\varepsilon)$ and associated to the pair (T,u). Then, there exists two times $0 < t_1 \leq t_2 < T$ s.t. $x([t_1,t_2]) \subset \bar{B}(0,\varepsilon)$. Since $0 < \varepsilon < R < \min\{r_0, r_f\}$, there exists $0 < t_{\text{in}} < t_1 \leq t_2 < t_{\text{out}} < T$ s.t. $x([t_{\text{in}}, t_{\text{out}}]) \subset \bar{B}(0,R)$ and $x(t_{\text{in}})$, $x(t_{\text{out}})$ belongs to $\partial \bar{B}(0,R) = S(0,R)$. Let consider the circular arc from $x(t_{\text{in}})$ to $x(t_{\text{out}})$ obtained with a control $v = (0,s)$, $s := \mathrm{sign}(\mu)$, and realized in a time $\tau > 0$. It is clear that $\tau \leq T_\theta(R,\mu)$ since $T_\theta(R,\mu)$ is the time to make a circular arc of angle 2π. It is also clear that $\tau \leq T_\theta(R,\mu) < T_r(\varepsilon, R, \mu)$ from lemma 1, and that $T_r(\varepsilon,R,\mu) < 2T_r(\varepsilon,R,\mu) \leq t_{\text{out}} - t_{\text{in}}$. Let us replace the part $x([t_{\text{in}}, t_{\text{out}}])$ by the circular arc. Then, the new trajectory associated to the pair denoted (T', u') is still admissible and is strictly better than x since $T' = T - (t_{\text{out}} - t_{\text{in}}) + \tau < T$. If the new trajectory is contained in $M \setminus B(0,\varepsilon)$, then it is finished, otherwise we repeat the process on the new trajectory. Note that the process may be repeated only a finite number of times since $T < +\infty$ and $T_r(\varepsilon, R, \mu) > 0$. ☐

Proof of theorem 1. By proposition 1, there exists an admissible trajectory x. Let T^* denote the first time s.t. $x(T^*) = x_f$. Let us introduce $R_1 := \varepsilon$ from lemma 2 and $R_2 := r_0 + T^*$, with $r_0 := \|x_0\|$. By lemma 2, the problem (P) is equivalent to the same problem with the additional constraint $R_1 \leq r(t)$. Since $\dot{r}(t) = v_1(t)$ and $v_1(t) \leq 1$, then for any $t \in [0, T^*]$ we have $r(t) \leq r_0 + T^*$. The problem (P) is thus equivalent to the same problem with the additional constraints: $R_1 \leq r(t) \leq R_2$. The trajectories of the equivalent problem are contained in the compact set $\bar{B}(0,R_2) \setminus B(0,R_1)$. The result follows from the Filippov's existence theorem. ☐

III. Pontryagin Maximum Principle

Let $(T,u) \in \mathscr{D}$ be an optimal solution of problem (P) and let $x := x_u(\cdot, x_0)$ denote the associated optimal trajectory. According to the Pontryagin Maximum Principle [7], there exists an absolutely continuous function $p : [0,T] \to \mathbb{R}^2$ satisfying the adjoint equation almost everywhere over $[0,T]$:

$$\dot{p}(t) = -\nabla_x H(x(t), p(t), u(t)), \tag{6}$$

where $H(x,p,u) := p \cdot (F_0(x) + u_1 F_1(x) + u_2 F_2(x))$ is the pseudo-Hamiltonian associated to (P).[1] Besides, $\exists\, p^0 \in \mathbb{R}$ s.t.:

$$p^0 \leq 0, \text{ the pair } (p(\cdot), p^0) \text{ never vanishes} \tag{7}$$

and the optimal control satisfies a.e. over $[0,T]$:

$$H(x(t), p(t), u(t)) = \max_{w \in \mathbf{U}} H(x(t), p(t), w) = -p^0. \tag{8}$$

Recall that an *extremal* $(x(\cdot), p(\cdot), p^0, u(\cdot))$ satisfying (4) and (6)–(8) is *abnormal* when $p^0 = 0$ and *normal* when $p^0 \neq 0$.

Definition 2. An extremal $(x(\cdot), p(\cdot), p^0, u(\cdot))$ is called a *BC-extremal* if $x(0) = x_0$ and there is a time $T \geq 0$ s.t. $x(T) = x_f$.

Let us introduce the Hamiltonian lifts $H_i(x,p) := p \cdot F_i(x)$, $i = 0,1,2$, the function $\Phi := (H_1, H_2)$ and the *switching function* φ which is defined for $t \in [0,T]$ by:

$$\varphi(t) := \Phi(z(t)) = p(t), \quad z(\cdot) := (x(\cdot), p(\cdot)).$$

[1]The standard inner product is written $a \cdot b$, for a, b in $\mathbb{R}^2$.

The maximization condition (8) implies for a.e. $t \in [0,T]$:

$$u(t) = \frac{\varphi(t)}{\|\varphi(t)\|} = \frac{p(t)}{\|p(t)\|},$$

whenever $\varphi(t) \neq 0$. Introducing the *switching locus*

$$\Sigma := \{z \in M \times \mathbb{R}^2;\ \Phi(z) = 0\} = M \times \{0\},$$

one can define, outside Σ, the Hamiltonian:

$$\mathbf{H}(z) := H_0(z) + \|\Phi(z)\| = H_0(z) + \|p\|.$$

Recall that a *switching time* $0 < t < T$ is a time s.t. $\varphi(t) = 0$ and s.t. for any $\varepsilon > 0$ (small enough) there exists a time $s \in (t-\varepsilon, t+\varepsilon) \subset [0,T]$ s.t. $\varphi(s) \neq 0$.

Definition 3. An extremal $(x(\cdot), p(\cdot), p^0, u(\cdot))$ contained outside the switching surface Σ is called of *order zero*.

Proposition 2. *All the extremals are of order zero, that is there are no switching times.*

Proof. Let $(x(\cdot), p(\cdot), p^0, u(\cdot))$ be an extremal. If there exists a time t s.t. $\varphi(t) = 0$, then $p(t) = 0$ and we have $H(x(t), p(t), u(t)) = 0 = -p^0$ which is impossible by (7). □

Denoting the drift $F_0(x,\mu)$ to emphasize the role of μ, one introduces for $(x,\mu) \in M \times \mathbb{R}$ the set

$$\mathscr{F}(x,\mu) := \{F_0(x,\mu) + \sum_{i=1}^{2} u_i F_i(x);\ u := (u_1,u_2) \in \mathbf{U}\}. \quad (9)$$

Then, we have

$$0 \in \mathscr{F}(x,\mu) \Leftrightarrow \exists u := (u_1,u_2) \in \mathbf{U} \text{ s.t. } F_0(x,\mu) = \sum_{i=1}^{2} u_i F_i(x)$$
$$\Leftrightarrow \|F_0(x,\mu)\| \leq 1 \Leftrightarrow |\mu| \leq \|x\|.$$

This leads to introduce the following definition.

Definition 4. The drift $F_0(x,\mu)$ is *weak* if $\|F_0(x,\mu)\| < 1$, *strong* if $\|F_0(x,\mu)\| > 1$ and *moderate* if $\|F_0(x,\mu)\| = 1$.

Proposition 3. *Let $(x(\cdot), p(\cdot), p^0, u(\cdot))$ be an abnormal extremal, that is $p^0 = 0$. Then, the drift is strong or moderate all along the trajectory.*

Proof. Since the extremal is of order 0, all along the extremal we have $\mathbf{H}(x(t), p(t)) = p(t) \cdot F_0(x(t),\mu) + \|p(t)\| = 0$. Hence, the Cauchy-Schwarz inequality gives

$$\|p(t)\| = |p(t) \cdot F_0(x(t),\mu)| \leq \|p(t)\| \|F_0(x,\mu)\|$$

and since $\|p(t)\| \neq 0$, the result follows. □

IV. Shooting function and second-order optimality conditions

A. Shooting function

For any differential equation of the form $\dot{z}(t) = f(z(t))$, we denote by $\exp(tf)(z_0)$ its solution at time t starting from z_0. For any Hamiltonian $\mathbf{H}(z)$, resp. pseudo-Hamiltonian $H(z,u)$, we denote by $\vec{\mathbf{H}}(z) := (\nabla_p \mathbf{H}(z), -\nabla_x \mathbf{H}(z))$, resp. $\vec{H}(z,u) := (\nabla_p H(z,u), -\nabla_x H(z,u))$, its associated *Hamiltonian system.*

Proposition 4. *Let $(x(\cdot), p(\cdot), p^0, u(\cdot))$ be an extremal. Denoting $z(\cdot) := (x(\cdot), p(\cdot))$, then, we have over $[0,T]$:*

$$\dot{z}(t) = \vec{H}(z(t), u(t)) = \vec{\mathbf{H}}(z(t)) = H_0'(z(t)) + \left(\frac{p(t)}{\|p(t)\|}, 0\right).$$

Proof. Since the extremal is of order zero, the control $t \mapsto u(t)$ is smooth and the adjoint equation (6) is satisfied all over $[0,T]$. Besides, denoting (with a slight abuse of notation) $u(z) := \Phi(z)/\|\Phi(z)\|$, we have:

$$\begin{aligned}\mathbf{H}'(z) &= \frac{\partial H}{\partial z}(z,u(z)) + \frac{\partial H}{\partial u}(z,u(z))\, u'(z) \\ &= \frac{\partial H}{\partial z}(z,u(z)) + \Phi(z)^T \Big(\frac{I_2}{\|\Phi(z)\|} - \frac{\Phi(z)\Phi(z)^T}{\|\Phi(z)\|^3}\Big)\, \Phi'(z) \\ &= \frac{\partial H}{\partial z}(z,u(z)) = \vec{H_0}(z) + \left(0, \frac{p}{\|p\|}\right).\end{aligned}$$

□

According to this proposition, one can define the *exponential mapping* by: $\exp_{x_0}(t,p_0) := \pi_x \circ \exp(t\vec{\mathbf{H}})(x_0,p_0)$, where $\pi_x(x,p) := x$ is the canonical projection, and the *shooting function* by:

$$S(T,p_0) := \left(\mathbf{H}(x_0,p_0) + p^0, \exp_{x_0}(T,p_0) - x_f\right),$$

where $(T,p_0) \in \mathbb{R} \times \mathbb{R}^2$ and where x_0 and p^0 are given. This formulation is equivalent to

$$\begin{array}{llll} S_P: & \mathbb{R} \times P & \longrightarrow & \mathbb{R}^2 \\ & (T,p_0) & \longmapsto & S_P(T,p_0) := \exp_{x_0}(T,p_0) - x_f. \end{array} \quad (10)$$

where $P := \{p_0 \in \mathbb{R}^2;\ \mathbf{H}(x_0,p_0) + p^0 = 0\}$.

Proposition 5. *Let $(x(\cdot), p(\cdot), p^0, u(\cdot))$ be a BC-extremal with $T \geq 0$ the first time s.t. $x(T) = x_f$. Then, $S(T, p(0)) = 0$. Conversely, let $(T,p_0) \in \mathbb{R}_+ \times \mathbb{R}^2$ s.t. $S(T,p_0) = 0$, with $x_0 \in M$ and $p^0 \leq 0$ given. Then, defining $u(t) := u(z(t))$ and $z(t) := \exp(t\vec{\mathbf{H}})(x_0,p_0)$ over $[0,T]$, the triplet $(z(\cdot), p^0, u(\cdot))$ is a BC-extremal.*

Remark 2. Since $\|p(t)\| \neq 0$, one can set $\|p(0)\| = 1$ instead of setting $p^0 = 0$ or $p^0 = -1$. Following this, one can restrict p_0 to the 1-sphere $\mathbb{S}^1$ and define the shooting function:

$$\begin{array}{llll} S_S: & \mathbb{R} \times \mathbb{S}^1 & \longrightarrow & \mathbb{R}^2 \\ & (T,p_0) & \longmapsto & S_S(T,p_0) := \exp_{x_0}(T,p_0) - x_f. \end{array} \quad (11)$$

If $(T,p_0) \in \mathbb{R}_+ \times \mathbb{S}^1$ satisfies $S_S(T,p_0) = 0$, then this pair is associated to a BC-extremal if and only if $\mathbf{H}(x_0,p_0) \geq 0$. Otherwise, the associated extremal is not *time-minimizing* but *time-maximizing*. This formulation has the advantage that p^0 is not imposed, so we can find abnormal and normal extremals with the same formulation, but it is required to check *a posteriori* the sign of $\mathbf{H}(x_0,p_0)$.

B. second-order optimality conditions

Since the extremals are of order zero, one can restrict $u(t)$ to the 1-sphere $\mathbb{S}^1$. Doing this, with some abuse of notations, we have $H = H_0 + u_1 H_1 + u_2 H_2 = H_0 + \cos\alpha H_1 + \sin\alpha H_2$, with α the new control. Differentiating twice, we have

$$\frac{\partial H}{\partial \alpha} = -\sin\alpha H_1 + \cos\alpha H_2,\ \frac{\partial^2 H}{\partial \alpha^2} = -(\cos\alpha H_1 + \sin\alpha H_2),$$

and since $u = (\cos\alpha, \sin\alpha) = \Phi/\|\Phi\|$, $\Phi = (H_1, H_2)$, we have

$$\frac{\partial^2 H}{\partial \alpha^2} = -\sqrt{H_1^2 + H_2^2} < 0$$

along any extremal. Hence, the *strict Legendre-Clebsch* condition is satisfied and we are in the *regular* case [1], [4], but with a *free* final time T. Let $(t_c, p_0) \in \mathbb{R}_+^* \times P$ and recall that t_c is a *conjugate time* if $\exp'_{x_0}(t_c, p_0) : \mathbb{R} \times T_{p_0}P \to \mathbb{R}^2$ is not invertible. If t_c is a conjugate time, then $\exp_{x_0}(t_c, p_0)$ is called a *conjugate point*. Introducing $F(x,u) := F_0(x) + u_1 F_1(x) + u_2 F_2(x)$ and assuming $F(x_0, u(x_0, p_0)) \neq 0$, then P is a regular submanifold of codimension 1 and $T_{p_0}P = \{v \in \mathbb{R}^2;\ F(x_0, u(x_0, p_0)) \cdot v = 0\}$.

Let $(x(\cdot), p(\cdot), p^0, u(\cdot))$ be an extremal. Then, t_c is a conjugate time if $\exists\,(\lambda, v) \in \mathbb{R} \times T_{p_0}P$, $p_0 := p(0)$, s.t.:

$$\begin{aligned}\exp'_{x_0}(t_c, p_0)\,(\lambda, v) &= \lambda \dot{x}(t_c) + \pi_x \circ \frac{\partial}{\partial p_0} \exp(t_c \vec{\mathbf{H}})(x_0, p_0)(v) \\ &=: \lambda \dot{x}(t_c) + \pi_x(\delta z(t_c)) \\ &=: \lambda \dot{x}(t_c) + \delta x(t_c) \\ &= 0.\end{aligned}$$

with $x_0 := x(0)$ and where we have introduced some notations. The function $\delta z(\cdot)$ is called a *Jacobi field* and is solution of the system

$$\dot{J}(t) = \vec{\mathbf{H}}'(z(t))\,J(t), \quad J(0) = (0, v),$$

denoting $z(\cdot) := (x(\cdot), p(\cdot))$. When the extremal has a conjugate time, then its local optimality (in a weak sense) is lost while the absence of conjugate times implies local optimality (in a strong sense). Classically, a conjugate time is computed solving:

$$t \mapsto \det(\delta x(t), \dot{x}(t)) = 0.$$

We refer to [1], [4] for more details.

Remark 3. If $(T, p_0) \in \mathbb{R}_+^* \times P$ satisfies $S_P(T, p_0) = 0$, see eq. (10), and if T is a conjugate time, then the Jacobian of the shooting function $S'_P(T, p_0)$ is not invertible. One can also use S_S, see eq. (11) instead of S_P and in this case, the vector v must satisfy $p_0 \cdot v = 0$ instead of $F(x_0, u(x_0, p_0)) \cdot v = 0$. This defines a second algorithm to compute conjugate times.

V. Applications

In the following examples, the initial condition is fixed to $x_0 = (2, 0)$ and we compute some BC-extremals for different final conditions and with different strengths for the drift. The HamPath code [2] is used to compute the BC-extremals and to check their local optimality.

*a) **HamPath code**:* A Newton-like algorithm is used to solve the shooting equation $S(T, p_0) = 0$. Providing $\mathbf{H}$ and S to HamPath, the code generates automatically the Jacobian of the shooting function. To make the implementation of S easier, HamPath supplies the exponential mapping. Automatic Differentiation is used to produce $\vec{\mathbf{H}}$ and is combined with Runge-Kutta integrators to assemble the exponential mapping and the variational equations (used to compute the Jacobi fields). See [2], [4] for more details about the implementation of the code and about the third-parties.

*b) **Example 1**:* For this first example we want to steer the particle from x_0 to $x_f = (-2, 0)$ with $\mu = 2\|x_0\|$ (strong drift). In this case, we obtain a final time $T \approx 1.641$ and the shooting equation $S = 0$, is solved with a very good accuracy (of order $1e^{-12}$, like the others examples). The associated normal trajectory is portrayed in Fig. 1. The point vortex is represented by a black dot while the initial condition is blue. The initial velocity $\dot{x}(0)$ is given with the boundary (the black circle) of $x_0 + \mathscr{F}(x_0, \mu)$, cf. eq. (9). One can see that the drift is strong since $x_0 \notin x_0 + \mathscr{F}(x_0, \mu)$. Besides, as shown in Fig. 2, $\det(\delta x(t), \dot{x}(t)) \neq 0$, $\forall t \in (0, T]$, so there are no conjugate times along this trajectory over $[0, T]$ and therefore the extremal is locally optimal, at least until T.

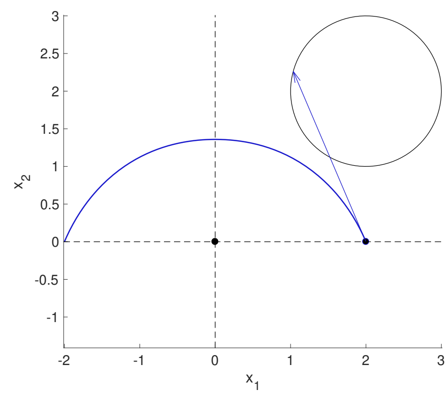

Fig. 1. Trajectory in a strong drift case with $\mu = 2\|x_0\|$, $x_0 = (2,0)$ and $x_f = (-2,0)$. The final time is $T \approx 1.641$.

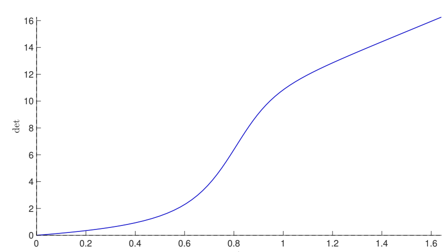

Fig. 2. Graph of $t \mapsto \det(\delta x(t), \dot{x}(t))$ over $[0, T]$ for the case of Fig. 1. As we can see, no conjugate time occurs.

*c) **Example 2**:* Now to emphazise the influence of the final condition let us take again $\mu = 2\|x_0\|$ and set $x_f = (2.5, 0)$. We can note from Fig. 3 that the solution turns around the point vortex and profits from the circulation. In this case we obtain a final time $T \approx 2.821$. This trajectory has no conjugate point neither.

*d) **Examples 3-4**:* Here we want to observe what happens for a weak drift. We set $\mu = 0.5\|x_0\|$ and presents two cases with $x_f = (-2, 0)$ (cf. Fig. 4) and $x_f = (2.5, 0)$ (cf. Fig. 5). When $x_f = (-2, 0)$, the final condition is the same as in the example 1 but since the drift is weaker, the final time is longer. This is because the particle takes advantage of the vortex circulation. On the other hand, for $x_f = (2.5, 0)$ (same final condition as example 2) and considering a weak drift, then the particle does not turn around the vortex, see Fig. 5. Here again, these trajectories has no conjugate points.

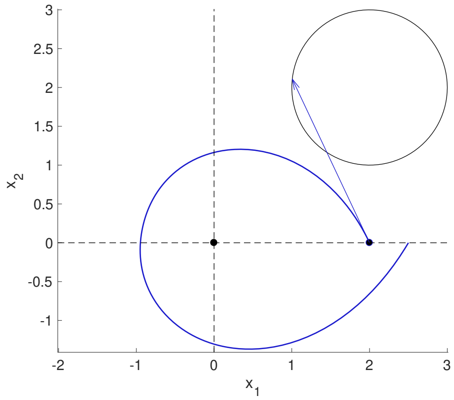

Fig. 3. Trajectory in a strong drift case with $\mu = 2\|x_0\|$, $x_0 = (2,0)$ and $x_f = (2.5,0)$. The final time is $T \approx 2.821$.

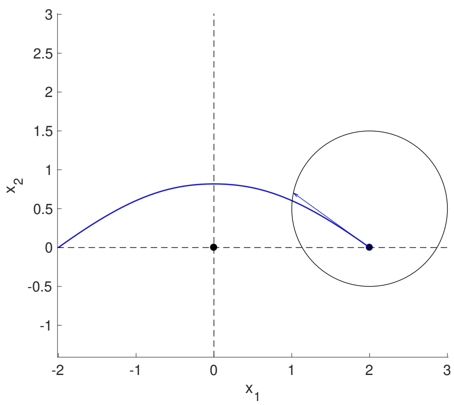

Fig. 4. Trajectory in a weak drift case with $\mu = 0.5\|x_0\|$, $x_0 = (2,0)$ and $x_f = (-2,0)$. The final time is $T \approx 2.826$. Compare to Fig. 1.

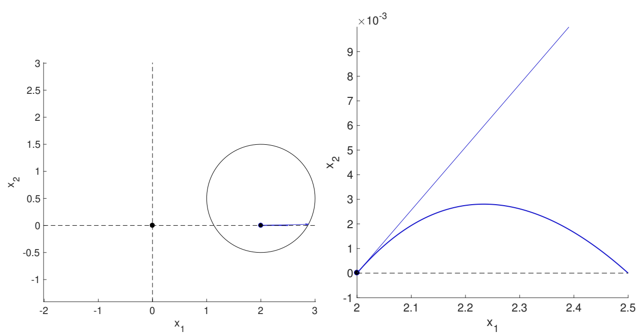

Fig. 5. Trajectory in a weak drift case with $\mu = 0.5\|x_0\|$, $x_0 = (2,0)$ and $x_f = (2.5,0)$. The final time is $T \approx 0.56$. Compare to Fig. 3. A zoom is given on the right-subgraph.

VI. CONCLUSIONS

In this work we have initiating the analysis of the minimal time control problem of passive tracers in interaction with point vortices. In the case of a single point vortex in a two-dimensional fluid, we have proved the controllability of the control system and the existence of time-minimal solutions. Thanks to the Pontryagin Maximum Principle, we have proved that the BC-extremals are only extremals of order zero (like in the Riemannian situation) and we gave two methods to compute them, analytically or numerically, thanks to the shooting functions. We have then computed different BC-extremals to illustrate the effects of the drift (weak or strong) and we have checked the local optimality of these extremals by the absence of conjugate times.

One objective of this work is to pave the road for the construction of the optimal synthesis. To compute the optimal synthesis, a first step is to compute the *conjugate locus*, that is the set of conjugate points (where the extremals ceases to be locally optimal). These examples suggest that for some extremals, there are no conjugate points. A step further would be to prove if the conjugate locus exists or not. A first thing to do is to compute the *curvature* [10] (in closed-form) along the extremals, since a nonpositive curvature implies the absence of conjugate points. If the curvature is sometimes positive, then the question remains. To complete the optimal synthesis, the next step is to compute the *cut locus*, that is the set of cut points (where the extremals ceases to be optimal). This is not an easy task, even in Riemannian geometry. The complexity in our situation comes from the Zermelo-like deformation, since in this case, the regularity of the value function depends on the strength of the drift. This has to be investigated.

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