About the prior-saturation phenomenon for minimal time problems in the plane

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***Abstract*— We consider minimal time problems governed by control-affine-systems in the plane. We focus on the synthesis problem in presence of a singular locus that involves a saturation point for the singular control. We show that the minimum time synthesis can exhibit a prior-saturation point at the intersection of the singular locus and a switching curve. We also provide a set of non-linear equations to compute the prior-saturation point, and, at this point, we show a tangency property involving the switching curve.**

I. INTRODUCTION

In this paper, we consider minimal time problems governed by control-affine-systems in the plane

$$\dot{x} = f(x) + u(t)g(x), \quad |u(t)| \leq 1,$$

where $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ are smooth vector fields. Syntheses for such problems have been investigated a lot in the literature [4], [5], [13], [16], [17], [18]. We focus here on the notion of *singular arc* which appears in the synthesis when the switching function (the scalar product between the adjoint vector and the controlled vector field g) vanishes over a time interval I. In that case, the corresponding singular control u_s (which allows the corresponding trajectory to stay on the singular locus) can be expressed in feedback form $x \mapsto u_s(x)$. However, it may happen that u_s becomes non admissible, *i.e.*, $x \mapsto |u_s(x)|$ takes values above the maximal value for the control. Such a situation naturally appears in several application models, see, *e.g.*, [1], [2], [10], [12]. In that case, we say that a *saturation phenomenon* occurs. This implies the following (non-intuitive) property that if a singular arc is optimal, then it should leave the singular locus at a so-called *prior-saturation point* before reaching the saturation point. This property has been studied in the literature in various situations such as for control-affine systems in dimension 2 and 4 (see, *e.g.*, [14], [15], [2] and references herein).

Our objective in this paper is to provide new qualitative properties on the minimum time synthesis in presence of a saturation point. More precisely, our objective is twofold:

- We first give a set of conditions on the system that ensure the occurence of the prior-saturation phenomenon, and we show that (under certain assumptions) the system leaves the singular arc at this point (before reaching the saturation point) with the maximal value for the control. This last arc is usually called *bridge* following the terminology as in [6], [7] (see also [3], [4]).
- Second, we introduce a shooting function that allows an effective computation of the prior-saturation point. In addition, we show that when the system exhibits a switching curve emanating from the prior-saturation point, then a tangency property occurs between this curve and the bridge.

The tangency property has been pointed out in several application models (see, *e.g.*, [2], [6]). To the best of our knowledge, this property has not been addressed previously in a general setting in the literature.

The paper is structured as follows: in Section II, we recall classical expressions and properties of singular controls for control-affine-systems in the plane introducing the saturation phenomenon. In Section III, we provide a set of conditions on the system that ensure occurrence of the prior-saturation phenomenon. In Section IV, we show a tangency property between the switching curve emanating from a prior-saturation point and the bridge, and we describe how to compute the prior-saturation point thanks to a shooting function constructed via the Hamiltonian lifts of f and g.

II. SATURATION PHENOMENON

The purpose of this section is to recall some facts about the minimum time control problems in the plane that will allow us to introduce the saturation phenomenon. Throughout the paper, $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^2$ associated with the standard inner product written $a \cdot b$ for $a, b \in \mathbb{R}^2$, and $a^\perp$ denotes the vector $a^\perp := (-a_2, a_1)$. The interior of a subset $S \subset \mathbb{R}^n$ is denoted by $\mathrm{Int}(S)$.

A. Pontryagin's Principle

We start by applying the classical optimality conditions provided by the Pontryagin Maximum Principle (PMP), see [11]. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be two vector fields of class C^2, and consider the controlled dynamics:

$$\dot{x} = f(x) + u(t)g(x), \tag{1}$$

with admissible controls in the set

$$\mathcal{U} := \{u : [0, +\infty) \to [-1, 1] \; ; \; u \text{ meas.}\}.$$

Given an initial point $x_0 \in \mathbb{R}^2$ and a non-empty closed subset $\mathcal{T} \subset \mathbb{R}^2$, we focus on the problem of driving (1) in minimal time from x_0 to the target set $\mathcal{T}$:

$$\inf_{u \in \mathcal{U}} T_u \quad \text{s.t. } x_u(T_u) \in \mathcal{T}, \tag{2}$$

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where $x_u(\cdot)$ denotes the unique solution of (1) associated with the control u such that $x_u(0) = x_0$ and $T_u \in [0, +\infty]$ is the first entry time of $x_u(\cdot)$ into the target set $\mathcal{T}$. We suppose hereafter that optimal trajectories exist[1] and we wish to apply the PMP on (2). The Hamiltonian associated with (2) is the function $H : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ defined as

$$H(x, p, p^0, u) := p \cdot f(x) + up \cdot g(x) + p^0.$$

If u is an optimal control and x_u is the associated trajectory steering x_0 to the target set $\mathcal{T}$ in time $T_u \geq 0$, the following conditions are fulfilled:

- There exist $p^0 \leq 0$ and an absolutely continuous function $p : [0, T_u] \to \mathbb{R}^2$ satisfying the adjoint equation: almost everywhere over $[0, T_u]$:

$$\dot{p}(t) = -\nabla_x H(x_u(t), p(t), p^0, u(t)). \tag{3}$$

- The pair $(p^0, p(\cdot))$ is non-zero.
- The optimal control u satisfies the *Hamiltonian condition* almost everywhere over $[0, T_u]$:

$$u(t) \in \mathrm{argmax}_{\omega \in [-1,1]} H(x_u(t), p(t), p^0, \omega). \tag{4}$$

- At the terminal time, the *transversality condition*[2] $p(T_u) \in -N_{\mathcal{T}}(x_u(T_u))$ is fulfilled.

Recall that an extremal $(x_u(\cdot), p(\cdot), p^0, u(\cdot))$ satisfying (1) and (3)-(4) is *abnormal* whenever $p^0 = 0$ and *normal* whenever $p^0 \neq 0$ (in the latter case, we take $p^0 = -1$ and the corresponding extremal is denoted by $(x_u(\cdot), p(\cdot), u(\cdot))$ and we shall then write $H(x, p, u)$ in place of $H(x, p, p^0, u)$). Since T_u is free and (1) is autonomous, the Hamiltonian H is zero along any extremal: for a.e. $t \in [0, T_u]$,

$$H = p(t) \cdot f(x_u(t)) + u(t)p(t) \cdot g(x_u(t)) + p^0 = 0. \tag{5}$$

The *switching function* ϕ is

$$\phi(t) := p(t) \cdot g(x_u(t)), \quad t \in [0, T_u], \tag{6}$$

and it gives us the following control law:

$$\begin{cases} \phi(t) > 0 & \Rightarrow \quad u(t) = +1, \\ \phi(t) < 0 & \Rightarrow \quad u(t) = -1, \\ \phi(t) = 0 & \Rightarrow \quad u(t) \in [-1, +1]. \end{cases} \tag{7}$$

A *switching time* is an instant $t_c \in (0, T_u)$ such that the control u is discontinuous at time t_c. We say that the corresponding extremal trajectory has a *switching point* at time t_c. Of particular interest is the case when there is a time interval $I := [t_1, t_2]$ such that the switching function vanishes over I, *i.e.*,

$$\phi(t) = p(t) \cdot g(x_u(t)) = 0, \quad t \in I.$$

We then say that the extremal trajectory has a *singular arc* over I. Note that we shall suppose such an extremal to be normal, *i.e.*, $p^0 \neq 0$. Indeed, recall from [5, Prop. 2 p.49] that under generic conditions, abnormal extremals are bang-bang. By differentiating ϕ w.r.t. t, one has

$$\dot{\phi}(t) = p(t) \cdot [f, g](x_u(t)), \quad t \in [0, T_u],$$

where $[f, g](x)$ is the Lie bracket of f and g at point x. The *singular locus* Δ_{SA} (in the state space) is then the (possibly empty) subset of $\mathbb{R}^2$ defined as:

$$\Delta_{SA} := \{x \in \mathbb{R}^2 \ ; \ \det(g(x), [f, g](x)) = 0\}. \tag{8}$$

For future reference, we set

$$\delta_{SA}(x) := \det(g(x), [f, g](x)), \quad x \in \mathbb{R}^2.$$

Note that if an extremal is singular over an interval $[t_1, t_2]$, then $x_u(t) \in \Delta_{SA}$ for any $t \in [t_1, t_2]$ because $p(\cdot)$ must be non-zero and orthogonal to $\mathrm{span}\{g(x_u(t)), [f, g](x_u(t))\}$ for $t \in [t_1, t_2]$. The *singular control* u_s is then the value of the control for which the trajectory stays on the singular locus Δ_{SA}. Differentiating $\dot{\phi}$ w.r.t. t gives:

$$\ddot{\phi}(t) = p(t) \cdot [f, [f, g]](x_u(t)) + u(t)p(t) \cdot [g, [f, g]](x_u(t)),$$

for a.e. $t \in [0, T_u]$. Therefore, u_s becomes:

$$u_s(t) := -\frac{p(t) \cdot [f, [f, g]](x_u(t))}{p(t) \cdot [g, [f, g]](x_u(t))}, \quad t \in [0, T_u], \tag{9}$$

provided that $p(t) \cdot [g, [f, g]](x_u(t))$ is non zero for $t \in [t_1, t_2]$. This expression of the singular control does not guarantee that u_s is admissible, that is, $u_s(t) \in [-1, 1]$:

- When $u_s(t) \in [-1, 1]$, the point $x_u(t)$ is said *hyperbolic* if $p(t) \cdot [g, [f, g]](x_u(t)) > 0$, and *elliptic* if $p(t) \cdot [g, [f, g]](x_u(t)) < 0$ (see [3]).
- When $|u_s(t)| > 1$ for some instant t, we say that a *saturation phenomenon* occurs and that the corresponding points of the singular locus are *parabolic* (see [3]).

Our purpose in what follows is precisely to investigate properties of the synthesis when saturation occurs.

B. Singular control and saturation phenomenon

In this part, we derive classical expressions of the singular control in terms of feedback that will allow us to introduce saturation points (in terms of the data defining the system). The *collinearity set* associated with (1) is the (possibly empty) subset of $\mathbb{R}^2$ defined as

$$\Delta_0 := \{x \in \mathbb{R}^2 \ ; \ \det(f(x), g(x)) = 0\}. \tag{10}$$

Consider now the functions $\delta_0, \psi : \mathbb{R}^2 \to \mathbb{R}$ defined respectively by $\delta_0(x) := \det(f(x), g(x))$ and

$$\psi(x) := -\frac{\det(g(x), [f, [f, g]](x))}{\det(g(x), [g, [f, g]](x))}. \tag{11}$$

Lemma 2.1: Suppose that $\Delta_{SA} \neq \emptyset$, that $x \mapsto \det(g(x), [g, [f, g]](x))$ is non-zero over Δ_{SA}, and consider a singular arc defined over an interval $[t_1, t_2]$. Then, one has:

$$u_s(t) = \psi(x(t)), \quad t \in [t_1, t_2]. \tag{12}$$

where $x(\cdot)$ is the corresponding singular trajectory such that $x(t) \in \Delta_{SA}$ for $t \in [t_1, t_2]$.

[1] If the target can be reached from x_0 and if f, g have linear growth, then (2) admits an optimal solution, thanks to Filippov's Existence Theorem.

[2] Here, $N_{\mathcal{T}}(x)$ stands for the (Mordukovitch) limiting normal cone to $\mathcal{T}$ at point $x \in \mathcal{T}$, which coincides with the normal cone in the sense of convex analysis when $\mathcal{T}$ is convex, see [19].

Proof: Along I, the adjoint vector can be expressed as $p(t) = -\frac{g^{\perp}(x(t))}{\delta_0(x(t))}$, $t \in [t_1, t_2]$. Since for $t \in [t_1, t_2]$, $\{f(x(t)), g(x(t))\}$ is a basis of $\mathbb{R}^2$, we deduce (decomposing $[f,[f,g]](x(t))$ and $[g,[f,g]](x(t))$ on this basis):

$$p(t)\cdot[f,[f,g]](x(t)) = \det(g(x(t)), [f,[f,g]](x(t)))\Lambda(x(t)),$$
$$p(t)\cdot[g,[f,g]](x(t)) = \det(g(x(t)), [g,[f,g]](x(t)))\Lambda(x(t)),$$

where $\Lambda(x) := \frac{g^{\perp}(x)\cdot f(x)}{\delta_0(x)^2}$, $x \notin \Delta_0$. Moreover, $g^{\perp}(x(t))\cdot f(x(t)) = -\delta_0(x(t))$ and thus, this scalar product is non-zero because $p^0 \neq 0$. This ends the proof. ∎

Remark 2.1: Steady-state singular points are defined as the points $x^\star \in \Delta_{SA} \cap \Delta_0$ such that $g(x^\star) \neq 0$, see [5] (if $\Delta_{SA} \cap \Delta_0 \neq \emptyset$). Such points are equilibria of (1) with $u = \psi(x)$. A singular arc defined over a time interval $[t_1, t_2]$ does not contain such a point because $f(x(t))$ and $g(x(t))$ must be linearly independent over $[t_1, t_2]$. But, it can contain points $x^\star \in \Delta_0 \cap \Delta_{SA}$ such that $g(x^\star) = 0$.

To introduce the notion of saturation point, it is convenient to consider a parametrization of Δ_{SA} as follows. When $\Delta_{SA} \cap \Delta_0$ is non-empty, $\Delta_{SA}\backslash\Delta_0$ may consist of several parts (or components), and we write this set as

$$\Delta_{SA}\backslash\Delta_0 = \bigcup_{j\in K} \gamma_j,$$

where K is an index set.

Lemma 2.2: Suppose that Δ_{SA} is non-empty and that $x \mapsto \det(g(x), [g,[f,g]](x))$ is non-zero over Δ_{SA}. Then, each component γ of $\Delta_{SA}\backslash\Delta_0$ can be parametrized by a one-to-one parametrization $\zeta : J \to \gamma$, $\tau \mapsto \zeta(\tau)$ of class C^1, where J is an interval of $\mathbb{R}$.

Proof: For $x \in \Delta_{SA}\backslash\Delta_0$, one has $\mathrm{span}\{f(x), g(x)\} = \mathbb{R}^2$. Hence, there exist $\alpha(x), \beta(x) \in \mathbb{R}$ such that

$$[f,g](x) = \alpha(x)f(x) + \beta(x)g(x). \tag{13}$$

By taking the determinant, we find that for $x \in \Delta_{SA}\backslash\Delta_0$,

$$\alpha(x) = -\frac{\det(g(x), [f,g](x))}{\delta_0(x)},\ \beta(x) = \frac{\det(f(x), [f,g](x))}{\delta_0(x)}.$$

Consider now a component γ of $\Delta_{SA}\backslash\Delta_0$ and $x \in \gamma$. By computing $[f,[f,g]](x)$ thanks to (13), we get

$$\det(g(x), [g,[f,g]](x)) = -\delta_0(x)\nabla\alpha(x)\cdot g(x), \quad x \in \gamma.$$

Since $x \mapsto \det(g(x), [g,[f,g]](x))$ is non-zero over Δ_{SA}, the preceding equality implies that the scalar product $\nabla\alpha(x)\cdot g(x)$ is non-zero. On the other hand, γ is defined by the implicit equation $\delta_{SA}(x) = 0$. Observe that for $x \notin \Delta_0$, $\delta_{SA}(x) = -\alpha(x)\delta_0(x)$. By taking the derivative, we find that for $x \notin \Delta_0$, one has $\nabla\delta_{SA}(x) = -\delta_0(x)\nabla\alpha(x) - \alpha(x)\nabla\delta_0(x)$. Therefore, for $x \in \gamma$, we obtain $\nabla\delta_{SA}(x) = -\delta_0(x)\nabla\alpha(x)$. We can conclude that for any point $x \in \gamma$, the derivative $\partial_1\alpha(x)$ or $\partial_2\alpha(x)$ is non-zero. We are then in a position to apply the implicit function theorem to δ_{SA} locally at each point $x \in \gamma$, which then implies the desired property. ∎

Under the assumptions of Lemma 2.2, given a component γ of Δ_{SA}, there is a parametrization ζ such that

$$\gamma := \{\zeta(\tau)\ ;\ \tau \in J\},$$

where $\zeta : J \to \mathbb{R}^2$ is C^1-mapping (injective) and J is an interval.

Definition 2.1: A point $x^* := \zeta(\tau^*)$ with $\tau^* \in \mathrm{Int}(J)$ is called *saturation point* if $\psi(x^*) = 1$, $\psi(\zeta(\tau)) \in (-1, 1)$ for any $\tau \in J$ such that $\tau < \tau^*$, and $\psi(\zeta(\tau)) > 1$ for any $\tau \in J$ such that $\tau > \tau^*$.

As well, we can define saturation points $x^\star$ such that $\psi(x^\star) = -1$, that is, when the lower bound of the admissible control set is saturated. Our next aim is to study the optimality of singular arcs in presence of a saturation point.

III. Existence of a prior-saturation point

In this section, we show that a prior-saturation phenomenon can occur whenever the system exhibits a saturation point. We start by introducing our main assumptions.

Assumption 3.1: System (1) satisfies the following:

(i) One has $\Delta_0 = \emptyset$ and $\delta_0(x) < 0$ for all $x \in \mathbb{R}^2$.

(ii) The set Δ_{SA} is non-empty, simply connected, and has exactly one saturation point x^* with $\psi(x^*) = 1$.

(iii) Along the singular locus, the strict Legendre-Clebsch optimality condition is satisfied, that is, any singular extremal $(x_u(\cdot), p(\cdot), u(\cdot))$ defined over $[t_1, t_2]$ satisfies:

$$\frac{\partial}{\partial u}\frac{d^2 H_u}{dt^2}(x_u(t), p(t), u(t)) > 0, \quad \forall t \in [t_1, t_2]. \tag{14}$$

(iv) If Γ_- is the forward semi-orbit of (1) with $u = -1$ with the initial condition x^* at time 0, then

$$\mathcal{T} \cap \Gamma_- = \emptyset. \tag{15}$$

(v) The target $\mathcal{T}$ is reachable from every point $x_0 \in \mathbb{R}^2$.

Remark 3.1: (i) The hypothesis $\Delta_0 = \emptyset$ is not restrictive since we could restrict our analysis to a component γ of Δ_{SA} in place of Δ_{SA}.
(ii) By the previous computations, we can observe that (14) is equivalent to

$$\det(g(x), [g,[f,g]](x)) > 0, \quad \forall x \in \Delta_{SA}.$$

Recall that, under the strict Legendre-Clebsch condition, the singular arc is a *turnpike*, *i.e.*, it is time-minimizing in every neighborhood of a hyperbolic point of Δ_{SA}, [3]. This property can be retrieved by the clock form argument [8].

Under Assumption 3.1, the singular locus Δ_{SA} is written $\Delta_{SA} := \zeta(J)$ where $J \subset \mathbb{R}$ is an interval and $\zeta : J \to \Delta_{SA}$ is a C^1-mapping. In addition, Δ_{SA} partitions the state space into two simply connected subsets $\Delta_{SA}^{\pm}$:

$$\Delta_{SA}^{+} := \{x \in \mathbb{R}^2\ ;\ \det(g(x), [f,g](x)) > 0\},$$
$$\Delta_{SA}^{-} := \{x \in \mathbb{R}^2\ ;\ \det(g(x), [f,g](x)) < 0\}.$$

Given a normal extremal $(x_u(\cdot), p(\cdot), u(\cdot))$, the function

$$t \mapsto \gamma_u(t) := \beta(x_u(t)) - \alpha(x_u(t))u(t), \quad t \in [0, T_u],$$

is well-defined since $\Delta_0 = \emptyset$.

Lemma 3.1: Along a normal extremal $(x_u(\cdot), p(\cdot), u(\cdot))$, the switching function ϕ satisfies the ODE

$$\dot{\phi}(t) = \gamma_u(t)\phi(t) + \alpha(x_u(t)) \quad \text{a.e. } t \in [0, T_u]. \tag{16}$$

Proof: The proof follows using the expression of $\dot{\phi}$ and the fact that the Hamiltonian H is constant equal to zero. ∎

The next proposition shows that an extremal containing a singular arc until the point $x^\star$ is not optimal.

Proposition 3.1: Suppose that Assumption 3.1 holds true, and consider an optimal trajectory steering x_0 to the target $\mathcal{T}$ in time T_u. Then, the corresponding extremal $(x_u(\cdot), p(\cdot), u(\cdot))$ does not contain a singular arc defined over a time interval $[t_1, t_2]$ such that $x_u(t_2) = x^*$.

Proof: Suppose by contradiction that there is a time interval $[t_1, t_2]$ such that the trajectory is singular over $[t_1, t_2]$ and such that $x_u(t_2) = x^*$. We claim that, at time t_2, the vector $f(x_u(t_2)) + g(x_u(t_2))$ is tangent to Δ_{SA}. Indeed, it is enough to check that the vector $f(x^*) + g(x^*)$ is orthogonal to $\nabla\delta_{SA}(x^*) = -\delta_0(x^*)\nabla\alpha(x^*)$. As we have seen in the proof of Lemma 2.2, one has for $x \in \Delta_{SA}$:

$$\begin{array}{ll} \det(g(x), [g,[f,g]](x)) & = -\delta_0(x)\nabla\alpha(x)\cdot g(x), \\ \det(g(x), [f,[f,g]](x)) & = -\delta_0(x)\nabla\alpha(x)\cdot f(x). \end{array} \tag{17}$$

These equalities imply that

$$\begin{aligned} &-\delta_0(x^*)\nabla\alpha(x^*)\cdot(f(x^*) + g(x^*)) \\ &= \det(g(x^*), [g,[f,g]](x^*)) + \det(g(x^*), [f,[f,g]](x^*)). \end{aligned}$$

Since $\psi(x^*) = 1$, the right member of the above equality is zero which shows the claim. Note also that for $x \in \Delta_{SA}$, (17) also implies the equalities

$$\begin{aligned} &\nabla\delta_{SA}(x)\cdot(f(x) + g(x)) \\ &\qquad = \det(g(x), [g,[f,g]](x))(1 - \psi(x)), \\ &\nabla\delta_{SA}(x)\cdot(f(x) - g(x)) \\ &\qquad = \det(g(x), [g,[f,g]](x))(-1 - \psi(x)). \end{aligned}$$

Consider now the unique solution x_- of (1) with $u = -1$ starting from x^* at time t_2. This trajectory enters into the set Δ_{SA}^- for $t > t_2$, t close to t_2, because one has $\nabla\delta_{SA}(x^*)\cdot(f(x^*) - g(x^*)) < 0$. Going back to the optimal trajectory, there are now two possibilities for $x_u(\cdot)$: for $t > t_2$, t close to t_2, either $x_u(\cdot)$ enters into Δ_{SA}^+ or into Δ_{SA}^- (because the singular control becomes non admissible).

Suppose first that $x_u(\cdot)$ enters into Δ_{SA}^+. Then, there is $\varepsilon > 0$ such that one has $\alpha(x_u(t)) > 0$ for $t \in (t_2, t_2 + \varepsilon]$. It follows from (16) that one has $u = +1$ on this interval. But the velocity set being convex, we obtain a contradiction with the non-admissibility of the singular control at x^* (because x_- enters into Δ_{SA}^-). It follows that the optimal trajectory necessarily enters into the set Δ_{SA}^-. But then, since $\alpha < 0$ in Δ_{SA}^-, (16) implies that $u = -1$ in some time interval $(t_2, t_2 + \varepsilon]$.

From Assumption 3.1, the forward semi-orbit with $u = -1$ starting from x^* does not reach the target set. Hence, $x_u(\cdot)$ must have a switching point to $u = +1$ in Δ_{SA}^- or it must reach Δ_{SA} with the control $u = -1$. We see from (16) that the first case is not possible because at a switching time t_c such that $x_u(t_c) \in \Delta_{SA}^-$, we would have $\dot{\phi}(t_c) \geq 0$ in contradiction with $\alpha(x_u(t_c)) < 0$.

Suppose now that $x_u(\cdot)$ reaches Δ_{SA} at some point $x := \zeta(\tau)$ with $\tau < \tau^*$. Then, we obtain $\nabla\delta_{SA}(x)\cdot(f(x) - g(x)) < 0$ since $\psi(x) > -1$. But, this contradicts the fact that $x_u(\cdot)$ reaches Δ_{SA} with $u = -1$ at point x (indeed, because at this point, the singular control is admissible, one must have $\nabla\delta_{SA}(x)\cdot(f(x) - g(x)) \geq 0$). In the same way, the trajectory cannot reach a point $x \in \Delta_{SA}$ such that $x = \zeta(\tau)$ with $\tau > \tau^*$.

We can conclude that for any time $t \geq t_2$, one has $u(t) = -1$, but then, the optimal trajectory cannot reach the target set which is a contradiction (Assumption 3.1 (iv)). This concludes the proof. ∎

As an example, if x_0 belongs to the singular locus and is such that $x_0 := \zeta(\tau_0)$ with $\tau_0 < \tau^*$, and if in addition, the optimal trajectory starting from x_0 contains a singular arc, then the trajectory should leave the singular locus before reaching x^*. Let us insist on the fact that this property of leaving the singular locus before reaching x^* relies on the fact that the optimal trajectory should contain a singular arc. In the fed-batch model presented in [2], this property can be easily verified.

We now introduce the following definition (in line with [10], [14], [15]). Hereafter, the notation $\mathcal{S}_{[\tau_0', \tau_0]}$ denotes a singular arc passing through the points $\zeta(\tau_0')$ and $\zeta(\tau_0)$ with $\tau_0' \leq \tau_0 < \tau^*$.

Definition 3.1: Let $\tau_0 < \tau^*$. A point $x_e := \zeta(\tau_e) \in \Delta_{SA}$ with $\tau_0 < \tau_e < \tau^*$ is called a *prior-saturation point* if the singular arc $\mathcal{S}_{[\tau_0, \tau]}$ ceases to be optimal for $\tau \geq \tau_e$.

This definition makes sense only for initial conditions $\zeta(\tau_0)$ with $\tau_0 < \tau^*$ because for $\tau_0 \geq \tau^*$, optimal controls are not singular (since the singular control is non-admissible). We highlight the dependency of x_e w.r.t. the initial condition $\zeta(\tau_0) \in \Delta_{SA}$ as follows.

Proposition 3.2: Suppose that Assumption 3.1 holds true and that there are $\tau_1 < \tau_2 < \tau^*$ such that any optimal trajectory starting from $\zeta(\tau_0)$ with $\tau_0 \in [\tau_1, \tau_2)$ contains a singular arc $\mathcal{S}_{[\tau_0, \tau_2]}$. Then, for any initial condition $\tau_0 \in [\tau_1, \tau_2)$, one has $x_e = \zeta(\tau_e)$ with

$$\tau_e := \sup\{\tau \in J \text{ and } \mathcal{S}_{[\tau_1, \tau]} \text{ is optimal}\}. \tag{18}$$

Moreover, for any $\tau_0 \in (\tau_e, \tau^*]$ an optimal trajectory starting at $\zeta(\tau_0)$ leaves the singular locus at $\zeta(\tau_0)$.

Proof: Let $E := \{\tau \in J \ ; \ \mathcal{S}_{[\tau_1, \tau]} \text{ is optimal}\}$ and $F := \{\tau \in J \ ; \ \mathcal{S}_{[\tau_0, \tau]} \text{ is optimal}\}$ where $\tau_0 \in [\tau_1, \tau_2)$ is fixed. Take a point $\tau \in F$. Then, from our assumption, $\mathcal{S}_{[\tau_1, \tau]}$ is also optimal (by concatenation) which shows that $\tau \in E$. On the other hand, if $\tau \in E$, then $\mathcal{S}_{[\tau_0, \tau]}$ is optimal (as a sub-arc). It follows that $E = F$ and, in addition, since x_e is defined as the point such that $\mathcal{S}_{[\tau_1, \tau]}$ ceases to be optimal, we obtain (18).

Finally, for every $\tau_0 \in (\tau_e, \tau^*)$, a singular arc $\mathcal{S}_{[\tau_0,\tau_0']}$ with $\tau_0 < \tau_0' < \tau^*$ cannot be optimal, since otherwise, this would contradict the definition of τ_e. It follows that for every $\tau_0 \in (\tau_e, \tau^*]$, no singular arc occurs and thus $x_e = \zeta(\tau_0)$. ■

This property implies in particular that for some initial conditions in Δ_{SA} (*e.g.*, for $x_0 := \zeta(\tau_1)$), then the optimal path has a singular arc until x_e and a switching point at the prior-saturation point.

Remark 3.2: In addition to Assumption 3.1 (in particular (15)), if $\mathcal{T}$ is not reachable with the constant control $u = -1$ from those points of Δ_{SA} located between x_e and x^*, then the maximal value for the control $u = +1$ is locally optimal from x_e. In other words, the *bridge* (the last arc leaving Δ_{SA}) corresponds to $u = +1$. This can be proved by using similar arguments as for proving Proposition 3.2. Since the singular arc is a turnpike, this additionnal hypothesis also implies the existence of a switching curve emanating from x_e. Our next aim is precisely to investigate more into details geometric properties of optimal paths at the point x_e.

IV. Tangency property and prior-saturation

A. Introduction to the tangency property

1) Prior-saturation lift: To introduce this concept, let us start with an example. Let us consider a target $\mathcal{T} := \{x_f\}$, $x_f \in \mathbb{R}^2$, with an optimal trajectory of the form $\gamma_- \gamma_s \gamma_+$, where γ_-, γ_+ and γ_s are arcs, respectively, with control $u = -1$, $u = +1$ and $u = u_s$, where u_s is the singular control. The PMP gives first order optimality conditions satisfied by this extremal trajectory, that we can write as a system of nonlinear equations, the so-called *shooting equations*. To introduce this set of equations, we introduce some notation: we define $H_f(z) := p \cdot f(x)$ and $H_g(z) := p \cdot g(x)$, $z := (x, p)$, the Hamiltonian lifts of f and g. Any other Hamiltonian lift is defined like this. Define also the Hamiltonians $H_\pm := H_f \pm H_g$ and $H_s := H_f + u_s H_g$, where u_s is viewed as a function of z:

$$u_s(z) := -\frac{p \cdot [f,[f,g]](x)}{p \cdot [g,[f,g]](x)} = -\frac{H_{[f,[f,g]]}(z)}{H_{[g,[f,g]]}(z)}.$$

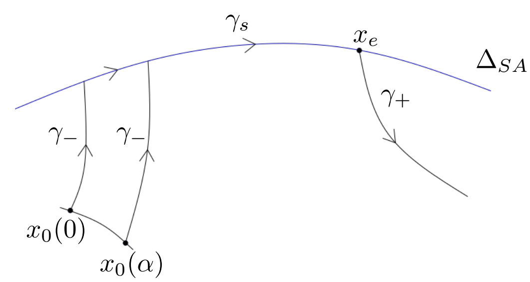
For any Hamiltonian H we define the *Hamiltonian system* $\overrightarrow{H} := (\partial_p H, -\partial_x H)$, and finally, we introduce the exponential mapping, such that the solution at time t of any differential equation $\dot{z}(s) = \varphi(z(s))$, with initial condition $z(0) = z_0$, is written $e^{t\varphi}(z_0)$ or $\exp(t\varphi)(z_0)$.

The shooting equations are then defined by the equation $S(y) = 0$, $y := (p_0, t_1, t_2, t_f, z_1, z_2) \in \mathbb{R}^{n+3+(2n)\times 2}$, $n = 2$, with

$$S(y) := \begin{pmatrix} H_g(z_1) \\ H_{[f,g]}(z_1) \\ H_+(\exp((t_f - t_2)\overrightarrow{H_+})(z_2)) + p^0 \\ \pi_x(\exp((t_f - t_2)\overrightarrow{H_+})(z_2)) - x_f \\ z_1 - \exp(t_1 \overrightarrow{H_-})(x_0, p_0) \\ z_2 - \exp((t_2 - t_1)\overrightarrow{H_s})(z_1) \end{pmatrix},$$

where $\pi_x(x, p) = x$, and $x_0 \in \mathbb{R}^2$, $p^0 \leq 0$ are given. The two first equations mean that the trajectory is entering the singular locus at z_1. Hence, the second arc is a singular arc. The third equation takes into account the free terminal time. The fourth equation implies that the last bang arc reaches the target $\mathcal{T} := \{x_f\}$ at the final time t_f, and the last two equations are the so-called *matching conditions* (which are not required but improve numerical stability). Now, let $y^* := (p_0, t_1, t_2, t_f, z_1, z_2)$ be a solution to $S(y) = 0$. Then, assuming $t_2 > 0$, the point $\pi_x(z_2)$ is a prior-saturation point and we call z_2 a *prior-saturation lift*.

2) The prior-saturation lift is locally unique: Let us consider a smooth and local one-parameter family of initial conditions $x_0(\alpha)$, $\alpha \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$. Let us assume that for any α, the optimal trajectory is of the form $\gamma_- \gamma_s \gamma_+$, and we denote by $y(\alpha)$ the corresponding solution of $S_\alpha(y)$, where S_α is defined as in section IV-A.1, but with initial condition $x_0(\alpha)$ in place of x_0. In addition, suppose that the lengths t_1, $t_2 - t_1$ and $t_f - t_2$ are positive, that is, each arc is defined on an interval with non-empty interior. Under this setting, for any α, we have $z_2(0) = z_2(\alpha)$, that is, the prior-saturation lift is locally unique. Hence, we can define $z_e := z_2(0)$ the prior-saturation lift. In relation with the property 3.2, we have $\pi_x(z_e) = x_e$, where x_e is the prior-saturation point. See the following illustration in the state space:



3) Computation of the prior-saturation lift: Actually, in the previous example, we can compute the prior-saturation lift with a smaller set of equations. The prior-saturation lift is the switching between the singular and positive bang arcs. What happens before is useless and we also do not need the matching equations. With these considerations we can set

$$F(t_b, z_b) := \begin{pmatrix} H_g(\exp(-t_b\overrightarrow{H_+})(z_b)) \\ H_{[f,g]}(\exp(-t_b\overrightarrow{H_+})(z_b)) \\ H_+(z_b) + p^0 \\ \pi_x(z_b) - x_f \end{pmatrix},$$

where $F : \mathbb{R}^5 \to \mathbb{R}^5$, and where we use the notation t_b, z_b (b stands for bridge) in relation with the concept of *bridge* defined in Ref. [6]. Note that the exponential mapping is here computed by backward integration, and with the notation of the two previous parts: we have $z_b = \exp((t_f - t_2)\overrightarrow{H_+})(z_2)$ and $t_b = t_f - t_2$. Hence, the prior-saturation lift is simply given by $z_e = \exp(-t_b\overrightarrow{H_+})(z_b)$.

From a general point of view, we can assume that the prior-saturation lift is given by solving a set of nonlinear equations of the following form:

$$F(t_b, z_b, \lambda) := \begin{pmatrix} H_{[f,g]}(\exp(-t_b\overrightarrow{H_+})(z_b)) \\ G(t_b, z_b, \lambda) \end{pmatrix}, \qquad (19)$$

where $\lambda \in \mathbb{R}^k$, $k \in \mathbb{N}$, represents some parameters, $F : \mathbb{R}^{5+k} \to \mathbb{R}^{5+k}$, and G is defined by

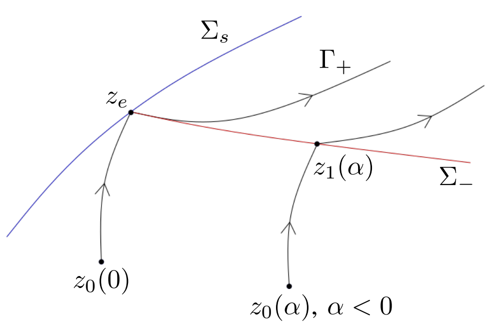
$$G(t_b, z_b, \lambda) := \begin{pmatrix} H_g(\exp(-t_b\overrightarrow{H_+})(z_b)) \\ H_+(z_b) + p^0 \\ \Psi(z_b, \lambda) \end{pmatrix}, \qquad (20)$$

with $\Psi(z_b, \lambda) \in \mathbb{R}^{2+k}$. It is important to notice that Ψ does not depend on t_b. In the previous example, we would have $\Psi(z_b) = \pi_x(z_b) - x_f$, that is, $k = 0$. For a more complex structure of the form $\gamma_-\gamma_s\gamma_+\gamma_-$, the parameter λ would be the last switching time between the γ_+ and γ_- arcs. In this case, Ψ would contain the additional switching condition $H_g = 0$ at this time.

4) The tangency property: Let us come back to the simple example where the solution is of the form $\gamma_-\gamma_s\gamma_+$. Let us consider a smooth and local one-parameter family of initial conditions $x_0(\alpha)$, $\alpha \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$. Let us also assume that for $\alpha = 0$, the optimal solution is of the form $\gamma_-\gamma_s\gamma_+$, but with γ_s reduced to a single point, that is, $t_2(0) - t_1(0) = 0$, with $y(0) := (p_0(0), t_1(0), t_2(0), t_f(0), z_1(0), z_2(0))$ the solution of the associated shooting equations. Assume also that for $\alpha > 0$, we are in the case of Sections IV-A.1 -IV-A.2, that is, $t_2(\alpha) - t_1(\alpha) > 0$, with $y(\alpha) := (p_0(\alpha), t_1(\alpha), t_2(\alpha), t_f(\alpha), z_1(\alpha), z_2(\alpha))$ the solution of the associated shooting equations. The prior-saturation lift is given by $z_e = z_2(\alpha)$ for $\alpha \in [0, \varepsilon)$. Let Γ_+ denotes the forward semi-orbit of $\dot{z} = \overrightarrow{H_+}(z)$ starting from z_e.

The idea is now to consider the case where there is a bifurcation in the structure of optimal trajectories when $\alpha = 0$. We thus assume that for $\alpha \in (-\varepsilon, 0)$, the solutions are of the form $\gamma_-\gamma_+$ and we denote by $z_1(\alpha)$ the switching point (in the cotangent space) between the two arcs. In this setting, there is a birth of a switching locus (still in the cotangent space) defined by $\Sigma_- := \{z_1(\alpha);\ \alpha \in (-\varepsilon, 0)\}$.

The aim of the next section is to prove that the semi-orbit Γ_+ is tangent to the switching curve Σ_- at the prior-saturation lift z_e. This is the tangency property. See the following illustration where $\Sigma_s : H_g = H_{[f,g]} = 0$ is the singular locus and where we define $z_0(\alpha) := (x_0(\alpha), p_0(\alpha))$:



B. Results

We start by a general definition of a prior saturation lift.

Definition 4.1: Let $(t_b^*, z_b^*, \lambda^*) \in \mathbb{R}^{5+k}$ be a solution to the equation $F = 0$ (recall (19)) and define $z_e := \exp(-t_b^*\overrightarrow{H_+})(z_b^*) \in \Sigma_s$. The point z_e is called a *prior-saturation lift* if $\pi_x(z_e)$ is a prior-saturation point.

Next, we introduce the following assumptions.

Assumption 4.1: The point z_e satisfies $u_s(z_e) < 1$.

Assumption 4.2: The function G from eq. (20) satisfies:

$$\begin{bmatrix} \dfrac{\partial G}{\partial z_b}(t_b^*, z_b^*, \lambda^*) & \dfrac{\partial G}{\partial \lambda}(t_b^*, z_b^*, \lambda^*) \end{bmatrix} \quad \text{invertible.}$$

Proposition 4.1: Suppose Assumptions 4.1 and 4.2 hold true, then $F'(t_b^*, z_b^*, \lambda^*)$ is invertible.

Proof: The Jacobian of F at $(t_b^*, z_b^*, \lambda^*)$ is given by:

$$F'(t_b^*, z_b^*, \lambda^*) = \begin{bmatrix} -a & * & * \\ -b & \dfrac{\partial G}{\partial z_b}(t_b^*, z_b^*, \lambda^*) & \dfrac{\partial G}{\partial \lambda}(t_b^*, z_b^*, \lambda^*) \end{bmatrix},$$

where $b = (H_{[f,g]}(z_e), 0, 0) = 0$ since $F(t_b^*, z_b^*, \lambda^*) = 0$ and $a = H_{[f,[f,g]]}(z_e) + H_{[g,[f,g]]}(z_e) \neq 0$ since $u_s(z_e) < 1$. ■

Note that the point z_e is locally unique by the inverse function theorem. From Assumption 3.1 and by Proposition 3.2, there exists a prior-saturation point x_e that is locally unique, and from Assumption 4.2, x_e has a locally unique lift in the cotangent space, given by the solutions to $F = 0$.

Lemma 4.1: Suppose that Assumption 4.2 holds true. Then, there is a curve, solution to $G = 0$, given by the graph of an implicit function $t_b \mapsto \sigma(t_b) := (z_b(t_b), \lambda(t_b)) \in \mathbb{R}^{4+k}$ satisfying $\sigma(t_b^*) = (z_b^*, \lambda^*)$, defined over an interval of the form $(t_b^* - \varepsilon, t_b^* + \varepsilon)$, $\varepsilon > 0$, and such that[3] $\sigma'(t_b^*) = 0_{\mathbb{R}^{4+k}}$.

Proof: The existence of σ is a simple application of the implicit function theorem. Its derivative is given by:

$$\sigma'(t_b) = -\begin{bmatrix} \dfrac{\partial G}{\partial z_b}[t_b] & \dfrac{\partial G}{\partial \lambda}[t_b] \end{bmatrix}^{-1} \cdot \frac{\partial G}{\partial t_b}[t_b].$$

where $[t_b]$ stands for $(t_b, \sigma(t_b))$. Since

$$\frac{\partial G}{\partial t_b}[t_b^*] = (H_{[f,g]}(z_e), 0_{\mathbb{R}^{3+k}}) = 0_{\mathbb{R}^{4+k}},$$

the result follows. ■

Definition 4.2: We define the *switching curve*

$$\Sigma := \{\varphi(t_b);\ t_b \in (t_b^* - \varepsilon, t_b^* + \varepsilon)\}$$

with $\varphi(t_b) := \exp(-t_b\overrightarrow{H_+})(z_b(t_b))$, and Γ_+ as the forward semi-orbit of $\dot{z} = \overrightarrow{H_+}(z)$ starting from z_e.

The curve Σ is called a switching curve since $H_g(\varphi(t_b)) = 0$. However, the switching curve is not necessarily optimal, that is, the optimal synthesis, with respect to the initial condition, may not contain Σ. Let us stratify Σ according to $\Sigma = \Sigma_- \cup \Sigma_0 \cup \Sigma_+$, with

$$\begin{aligned} \Sigma_- &:= \{\varphi(t_b);\ t_b \in (t_b^* - \varepsilon, 0)\}, \\ \Sigma_0 &:= \{\varphi(t_b^*)\} = \{z_e\}, \\ \Sigma_+ &:= \{\varphi(t_b);\ t_b \in (0, t_b^* + \varepsilon)\}. \end{aligned}$$

A typical situation is the following: $\Sigma_- \cup \Sigma_0$ is contained in the optimal synthesis while Σ_+ is not optimal for local and/or global optimality reasons. Our main result is the following.

Theorem 4.1: Suppose that Assumptions 3.1 and 4.2 hold true. Then, there exist a prior-saturation lift z_e and a switching curve Σ which is tangent to the semi-orbit Γ_+ at z_e.

[3] We assume that all the functions are smooth enough.

Proof: From Assumption 3.1, there exists a prior-saturation point. Since the system F from eq. (19) is assumed to be obtained from the PMP, the prior-saturation lift $z_e := \exp(-t_b^* \overrightarrow{H_+})(z_b^*)$ is well defined. From Assumption 4.2, z_e is locally unique (since $u_s(z_e) < 1$) and we can define Σ and Γ_+ according to Definition 4.2. To prove the tangency property, we have to show that $\varphi'(t_b^*)$ is colinear to $\overrightarrow{H_+}(z_e)$. For any $t_b \in (t_b^* - \varepsilon, t_b^* + \varepsilon)$, we have

$$\varphi'(t_b) = -\overrightarrow{H_+}(\varphi(t_b)) + \Phi(t_b, z_b(t_b)) \cdot z_b'(t_b),$$

where $\Phi(t, z_0)$ is defined as the solution at time t of the Cauchy problem $\dot{X}(s) = A(s, z_0)X(s)$, $X(0) = I$, with

$$A(s, z_0) := -\overrightarrow{H_+}'(\exp(-s\overrightarrow{H_+})(z_0)).$$

By lemma 4.1, $z_b'(t_b^*) = 0$ whence the result. ■

Let $\xi(z) := (H_g(z), H_{[f,g]}(z))$, then $\Sigma_s = \xi^{-1}(\{0_{\mathbb{R}^2}\})$.

Proposition 4.2: Suppose that ξ is a submersion at z_e and that Assumptions 4.1 and 4.2 hold true. Then the switching curve Σ is transverse to the singular locus Σ_s at z_e.

Proof: Since ξ is a submersion at z_e, then Σ_s is locally a regular submanifold of codimension two near z_e. Its tangent space at z_e is given by the kernel of $\xi'(z_e)$. But

$$\begin{aligned}\xi'(z_e) \cdot \varphi'(t_b^*) &= -\xi'(z_e) \cdot \overrightarrow{H_+}(z_e) \\ &= -(H_{[f,g]}(z_e), H_{[f,[f,g]]}(z_e) + H_{[g,[f,g]]}(z_e)) \\ &\neq (0, 0),\end{aligned}$$

since z_e is prior-saturation lift. ■

V. Conclusion

In presence of a saturation phenomenon of the singular control, the tangency property is a useful information for the computation of an optimal synthesis and optimal paths near the prior-saturation point. It also appears in other settings such as in Lagrange control problems governed by one-dimensional systems, see, *e.g.*, [9]. Future works could then investigate the prior-saturation phenomenon and the tangency property in other frameworks or in dimension $n \geq 3$.

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