Time-optimal aircraft trajectories in climbing phase and singular perturbations

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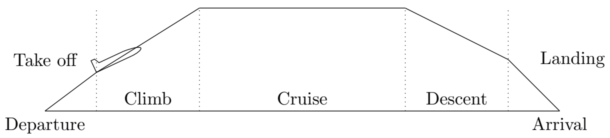
Abstract: This article deals with the singularly perturbed aircraft minimum time-to-climb problem. First, we introduce a reduced-order problem with affine dynamics with respect to the control and analyze it with the tools from geometric control: maximum principle combined with second order optimality conditions. Then, we compute candidates as minimizers using multiple shooting and homotopy methods for both problems, that we briefly compare numerically. Particular attention is paid to the singular extremals and we discuss about their local optimality.

Keywords: Aircraft trajectory, geometric optimal control, singular perturbation.

1. INTRODUCTION

A flight is composed of several phases which are take-off, climb, cruise, descent, approach and landing.



In this article, we are interested in the time optimal control of an aircraft during its climbing phase. This phase is determined by its own dynamics governed by the four-dimensional system

$$\frac{\mathrm{d}h}{\mathrm{d}t} = v\,\sin\gamma \tag{1}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{T(h)}{m} - \frac{1}{2}\frac{\rho(h)Sv^2}{m}\left(C_{D_1} + C_{D_2}u^2\right) - g\,\sin\gamma \tag{2}$$

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -C_s(v)\,T(h) \tag{3}$$

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{1}{2}\frac{\rho(h)Sv}{m}\,u - \frac{g}{v}\cos\gamma, \tag{4}$$

where the state variable $x := (h,\, v,\, m,\, \gamma)$ is composed of the altitude, the true air speed, the weight and the air slope of the aircraft. In this model, the lift coefficient is taken as the control variable u. The thrust $T(h)$ and the fuel flow $C_s(v)$ are given by the *Base of Aircraft DAta* (BADA) model:

$$T(h) := C_{T_1}(1 - \frac{h}{C_{T_2}} + h^2 C_{T_3}),$$
$$C_s(v) := C_{s_1}(1 + \frac{v}{C_{s_2}}),$$

where the constants C_{T_i}, C_{s_i}, with C_{D_i} (drag coefficients) are specific to the aircraft. The *International Standard Atmospheric* model provides the expression of the pressure $P(h)$, the temperature $\Theta(h)$ and the air density $\rho(h)$:

$$P(h) := P_0\left(\frac{\Theta(h)}{\Theta_0}\right)^{\frac{g}{\beta R}},$$
$$\Theta(h) := \Theta_0 - \beta h,$$
$$\rho(h) := \frac{P(h)}{R\Theta(h)}.$$

The remaining data are positive constants: g is the gravitational constant, S the wing area, R the specific constant of air, β the thermical gradient and P_0, Θ_0, the pressure and the temperature at the sea level.

The underlying optimal control problem consists of minimizing the transfer time, *i.e.* the time to reach the cruise phase, with fixed initial and final states. It is well known that the dynamics (1)–(4) contains slow (the mass m) and fast (the air slope γ) variables [1], see Ardema (1976) and Nguyen (2006). This time scale separation is normally treated by a singular perturbation analysis by replacing the fast state equation with the following:

$$\varepsilon\,\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{1}{2}\frac{\rho(h)Sv}{m}\,u - \frac{g}{v}\cos\gamma, \quad \varepsilon > 0. \tag{4'}$$

Let (P_ε) denote [2] the time-to-climb problem with the additional artificial parameter $\varepsilon > 0$. From the control theory, for a fixed value of $\varepsilon > 0$, the candidates as minimizers are selected among a set of extremals, solution of a Hamiltonian system given by the Pontryagin Maximum Principle (PMP), see Pontryagin et al. (1962). The application of the PMP leads to define a Boundary Value Problem (BVP_ε) which can be solved using shooting methods combined with numerical tests to check second-order optimality conditions.

When dealing with singular perturbation, one classical approach consists in applying the PMP to problem (P_ε)

[1] The altitude h and the true air speed v are fast compare to the mass but slow compare to the air slope. In this article, we consider h and v as slow variables.
[2] See section 4 for the exact definition of (P_ε), $\varepsilon > 0$.

and then solve the associated singular boundary value problem with matching asymptotic expansion techniques, see Ardema (1976) and Moïsséev (1985). In section 3, to approximate the solutions of problem (P_ε), $\varepsilon > 0$, we consider a reduced-order problem denoted[3] (P_0), where ε is put to zero and where γ is taken as the control variable. This technique is also applied in Nguyen (2006) and is the first step of a second approach presented in Moïsséev (1985). The problem (P_0) has one state variable less than (P_ε), $\varepsilon > 0$, and can be tackled with the tools from geometric optimal control. This allow first to study the classification of the extremals, solution of the PMP, and then to check, on the singular arcs, second-order conditions of optimality.

We complete this analysis showing numerically in section 4 that problem (P_0) from section 3 is a good approximation of problem (P_ε) when $\varepsilon = 1$. We use the `HamPath` software, Caillau et al. (2011), combining multiple shooting techniques with homotopy methods to solve the family of boundary value problems (BVP_ε) for $\varepsilon \in [1\,,100]$, starting from the simpler problem when $\varepsilon = 100$.

2. GEOMETRIC OPTIMAL CONTROL

2.1 The time optimal control problem

One considers a time optimal control problem given by the following data. A control domain $U \subset \mathbb{R}^m$. A smooth control system:

$$\dot{x}(t) = F(x(t), u(t)), \tag{5}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in U$, $m \le n$. An initial state $x_0 \in \mathbb{R}^n$ and a final state $x_f \in \mathbb{R}^n$. The optimal control problem can be written

$$\min_{t_f>0,\, u(\cdot)\in\mathcal{U}_{t_f}} \left\{ t_f \mid E_{x_0,t_f}(u(\cdot)) = x_f \right\}, \tag{$P_{\rm tmin}$}$$

where $\mathcal{U}_{t_f} := \mathrm{L}^\infty([0\,,t_f], U)$ is the set of *admissible controls*[4] and where E_{x_0,t_f} is the *end-point mapping* defined by: $E_{x_0,t}\colon \mathcal{U}_t \to \mathbb{R}^n$, $E_{x_0,t}(u(\cdot)) := x(t, x_0, u(\cdot))$, where $t \mapsto x(t, x_0, u(\cdot))$ is the trajectory solution of (5) with control $u(\cdot)$ and such that $x(0, x_0, u(\cdot)) = x_0$.

2.2 Pontryagin Maximum Principle

Let define the pseudo-Hamiltonian:

$$\begin{aligned} H\colon \mathbb{R}^n \times (\mathbb{R}^n)^* \times U &\longrightarrow \mathbb{R} \\ (x, p, u) &\longmapsto H(x, p, u) := \langle p, F(x, u)\rangle\,. \end{aligned}$$

By virtue of the *maximum principle*, see Pontryagin et al. (1962), if $(\bar{u}(\cdot), \bar{t}_f)$ is solution of $(P_{\rm tmin})$ then the associated trajectory $\bar{x}(\cdot)$ is the projection of an absolutely continuous integral curve $(\bar{x}(\cdot), \bar{p}(\cdot))$ of $\boldsymbol{H} := (\partial_p H, -\partial_x H)$ such that the following *maximization condition* holds for almost every $t \in [0\,, \bar{t}_f]$:

$$H(\bar{x}(t), \bar{p}(t), \bar{u}(t)) = \max_{u\in U} H(\bar{x}(t), \bar{p}(t), u). \tag{6}$$

The following boundary conditions must be satisfied:

$$\bar{x}(\bar{t}_f) = x_f, \tag{7}$$

$$H(\bar{x}(\bar{t}_f), \bar{p}(\bar{t}_f), \bar{u}(\bar{t}_f)) = -p^0, \tag{8}$$

with $p^0 \le 0$ and $(\bar{p}(\cdot), p^0) \neq (0, 0)$.

Definition 1. We call *extremal* a triplet $(x(\cdot), p(\cdot), u(\cdot))$ where $(x(\cdot), p(\cdot))$ is an integral curve of $\boldsymbol{H}$ satisfying (6). It is called a *BC-extremal* if it satisfies also (7) and (8).

2.3 Preliminaries about singular trajectories

A complete presentation of singular extremals – which play a crucial role in our analysis – can be found in Bonnard and Chyba (2003). In the following definitions and proposition we assume $U = \mathbb{R}^m$, *i.e.* there is no control constraint.

Definition 2. A control $u(\cdot) \in \mathcal{U}_T$ is called *singular* on $[0\,,T]$ if the derivative of $E_{x_0,T}$ at $u(\cdot)$ is not surjective.

We have the following characterization of singular controls which allows a practical computation.

Proposition 3. The control $u(\cdot)$ with its associated trajectory $x(\cdot)$ are singular on $[0\,,T]$ if and only if there exists a non zero adjoint $p(\cdot)$ such that $(x(\cdot), p(\cdot), u(\cdot))$ is solution a.e. on $[0\,,T]$ of

$$\dot{x} = \partial_p H, \quad \dot{p} = -\partial_x H, \quad 0 = \partial_u H. \tag{9}$$

If $u(\cdot)$ is singular then its associated adjoint satisfies for each $0 < t \le T$, $p(t) \perp \mathrm{im}\, \mathrm{d}E_{x_0,t}(u(\cdot))$.

Definition 4. A singular extremal is a triple $(x(\cdot), p(\cdot), u(\cdot))$ solution of (9). It is called:

- *Regular* if $\partial^2_{uu} H$ is negative definite (strict Legendre-Clebsch condition).
- *Strongly normal* if $\mathrm{im}\, \mathrm{d}E_{x(t_1),t_2-t_1}(u(\cdot)|_{[t_1,t_2]})$ is of corank one for each $0 < t_1 < t_2 \le T$.
- *Exceptional* if $H = 0$ and *normal* if $H \neq 0$.

2.4 Singular trajectories in the regular case

In the regular case, using the implicit function theorem, we can solve locally the equation $\partial_u H = 0$ and compute the smooth singular control as a function $\bar{u}(z)$, $z := (x, p)$. Plugging such $\bar{u}$ in H defines a true Hamiltonian $h(z) := H(z, \bar{u}(z))$. One can define the *exponential mapping* $\exp_{x_0}\colon (t, p_0) \mapsto \pi_x(\exp(t\boldsymbol{h})(x_0, p_0))$, where $\boldsymbol{h} := (\partial_p h, -\partial_x h)$, x_0 is fixed, π_x is the $x-$projection $(x, p) \mapsto x$, and $\exp(t\boldsymbol{h})(x_0, p_0)$ is the extremal $z(\cdot)$ at time t solution of $\dot{z}(s) = \boldsymbol{h}(z(s))$, $s \in [0\,,t]$, $z(0) = (x_0, p_0)$. The domain of $\exp_{x_0}$ is $\mathbb{R}^+ \times P$ where $P := \left\{p_0 \in (\mathbb{R}^n)^* \mid h(x_0, p_0) = -p^0\right\}$. This leads to the following definition.

Definition 5. Let $z(\cdot) := (x(\cdot), p(\cdot))$ be a reference extremal solution of $\boldsymbol{h}$. The time t_c is said to be *geometrically conjugate* if $(t_c, p(0))$ is a critical point of $\exp_{x(0)}$.

We have the following standard test:

Proposition 6. The time t_c is geometrically conjugate if and only if there exists a Jacobi field $J(\cdot) := (\delta x(\cdot), \delta p(\cdot))$ solution of the variational equation $\delta\dot{z}(t) = \mathrm{d}\boldsymbol{h}(z(t))\cdot\delta z(t)$, vertical at 0 and t_c, *i.e.* $\delta x(0) = \delta x(t_c) = 0$, and such that $\delta x(\cdot) \not\equiv 0$ on $[0\,,t_c]$.

Let $\bar{z}(\cdot)$ be a reference extremal. If the maximized Hamiltonian, $z \mapsto \max_{u\in\mathbb{R}^m} H(z, u)$, is well defined and smooth in a neighbourhood of $\bar{z}(\cdot)$, then one necessarily has $h(z) = H(z, \bar{u}(z)) = \max_{u\in\mathbb{R}^m} H(z, u)$ on this neighbourhood under the Legendre-Clebsch condition.

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[3] See section 3 for the exact definition of (P_0).

[4] The set of admissible controls is the set of L^∞-mappings on $[0\,,t_f]$ taking their values in U such that the associated trajectory $x(\cdot)$ is globally defined on $[0\,,t_f]$.

A trajectory $\bar{x}(\cdot)$ is called *strictly $\mathcal{C}^0$-locally* optimal if it realizes a strict local minimum $\bar{t}_f$ of the cost t_f w.r.t. all trajectories of the system close to $\bar{x}(\cdot)$ in $\mathcal{C}^0([0\,,\bar{t}_f],\mathbb{R}^n)$ (endowed with the uniform topology) and having the same endpoints as $\bar{x}(\cdot)$. The following result from Agrachev and Sachkov (2004) is crucial in our optimality analysis.

Theorem 7. For a normal regular extremal defined on $[0\,,\bar{t}_f]$, in the neighbourhood of which the maximized Hamiltonian is smooth, the absence of conjugate time on $(0\,,\bar{t}_f]$ is sufficient for strict $\mathcal{C}^0$-local optimality.

Under the additional strong regularity assumption, the extremal is not locally optimal in L^∞ topology on $[0\,,t]$, for every $t > t_c$, with t_c the first conjugate time.

2.5 Singular trajectories in the case of affine systems

For optimality analysis, one restricts our study to a single input affine system: $\dot{x} = F_0(x) + u\, F_1(x)$, $|u| \leq 1$. Relaxing the control bound, singular trajectories are parameterized by the constrained Hamiltonian system:

$$\dot{x} = \partial_p H, \quad \dot{p} = -\partial_x H, \quad 0 = \partial_u H = H_1, \qquad (10)$$

with $H_1(x,p) := \langle p, F_1(x)\rangle$ the Hamiltonian lift of F_1. The singular extremals are not regular and the constraint $H_1 = 0$ has to be differentiated at least twice along an extremal to compute the control. Introducing the Lie brackets of two vector fields F_0 and F_1, computed with the convention $F_{01}(x) := [F_0, F_1](x) := \mathrm{d}F_1(x)\cdot F_0(x) - \mathrm{d}F_0(x)\cdot F_1(x)$, and related to the Poisson bracket of two Hamiltonian lifts H_0 and H_1 by the rule $H_{01} := \{H_0, H_1\} = H_{[F_0,F_1]}$, one gets:

$$H_1 = H_{01} = H_{001} + u\, H_{101} = 0.$$

A singular extremal along which $H_{101} \neq 0$ is called of minimal order and the corresponding control is given by:

$$u_s(z) := -\frac{H_{001}(z)}{H_{101}(z)}.$$

Plugging such u_s in H defines a true Hamiltonian, denoted h_s, whose solutions initiating from $H_1 = H_{01} = 0$ define the singular extremals of order 1. They are related to the regular case using the Goh transformation, see Bonnard and Chyba (2003). The Legendre-Clebsch condition is replaced and we have the following additional necessary condition of optimality deduced from the high-order maximum principle, see Krener (1977). If the control $u_s(\cdot)$ is singular and non saturating, *i.e.* $|u_s| < 1$, the following generalized Legendre-Clebsch condition must hold along the associated singular extremal:

$$\frac{\partial}{\partial u}\frac{\partial^2}{\partial t^2}\frac{\partial H}{\partial u} = H_{101} \geq 0. \qquad (11)$$

2.6 Generic classification of the bang-bang extremals

We consider the affine system $\dot{x} = F_0(x) + u\, F_1(x)$, $|u| \leq 1$. An important issue is to apply the results from Kupka (1987) to classify the extremal curves near the *switching surface*. The switching surface is the set $\Sigma\colon H_1 = 0$, while the *switching function* is $t \mapsto \Phi(t) := H_1(z(t))$, where $z(\cdot)$ is an extremal curve. Let $\Sigma_s\colon H_1 = H_{01} = 0$. The singular extremals are entirely contained in Σ_s. A *bang-bang* extremal $z(\cdot)$ on $[0\,,T]$ is an extremal curve with a finite number of switching times $0 \leq t_1 < \cdots < t_n \leq T$. We denote by σ_+, σ_- the bang arcs for which $u(\cdot) \equiv \pm 1$ and by σ_s a singular arc. We denote by $\sigma_1\sigma_2$ an arc σ_1 followed by an arc σ_2.

Ordinary switching time. It is a time t such that two bang arcs switch with $\Phi(t) = 0$ and $\dot{\Phi}(t) = H_{01}(z(t)) \neq 0$. According to the maximum principle, near Σ, the extremal is of the form $\sigma_-\sigma_+$ if $\dot{\Phi}(t) > 0$ and $\sigma_+\sigma_-$ if $\dot{\Phi}(t) < 0$.

Fold case. It is a time where a bang arc has a contact of order 2 with Σ. Denoting $\ddot{\Phi}_\pm(t) := H_{001}(z(t)) \pm H_{101}(z(t))$ the second derivative of Φ with $u(\cdot) \equiv \pm 1$, we have three cases (if $\ddot{\Phi}_\pm \neq 0$) depending on $\ddot{\Phi}_\pm$ at the switching time:

(1) *Hyperbolic case:* $\ddot{\Phi}_+ > 0$ and $\ddot{\Phi}_- < 0$. A connection with a singular extremal is possible at Σ_s and locally each extremal is of the form $\sigma_\pm\sigma_s\sigma_\pm$ (by convention each arc of the sequence can be empty).
(2) *Parabolic case:* $\ddot{\Phi}_+\ddot{\Phi}_- > 0$. The singular extremal at the switching point is not admissible and every extremal curve is locally bang-bang with at most two switchings, *i.e.* $\sigma_+\sigma_-\sigma_+$ or $\sigma_-\sigma_+\sigma_-$.
(3) *Elliptic case:* $\ddot{\Phi}_+ < 0$ and $\ddot{\Phi}_- > 0$. A connection with a singular arc is not possible and locally each extremal is bang-bang but with no uniform bound on the number of switchings.

2.7 Conjugate points in the affine case for singular arcs

This section relies on the work of Bonnard and Kupka (1993). Let $z(\cdot) := (x(\cdot), p(\cdot))$ be a reference singular extremal defined on $[0\,,T]$. In the affine case, a Jacobi field $J(\cdot) := (\delta x(\cdot), \delta p(\cdot))$ is a solution of the variational equation $\delta\dot{z}(t) = \mathrm{d}\boldsymbol{h_s}(z(t))\cdot\delta z(t)$, $t \in [0\,,T]$, satisfying also $\mathrm{d}H_1(z(t))\cdot J(t) = \mathrm{d}H_{01}(z(t))\cdot J(t) = 0$. The Jacobi field is said *semi-vertical* at time t if $\delta x(t) \in \mathbb{R}F_1(x(t))$. A time $t_c \in (0\,,T]$ is a conjugate time if and only if there exists a non trivial Jacobi field semi-vertical at 0 and t_c.

We introduce the following assumptions:

(A1) $H_{101} \neq 0$ along $z(\cdot)$, F_0 and F_1 are linearly independent along $x(\cdot)$ and $x(\cdot)$ is injective.
(A2) $K(t) := \mathrm{Span}\left\{\mathrm{ad}^k F_0 \cdot F_1(x(t)) \mid k = 0, \cdots, n-2\right\}$ has codimension one, where $\mathrm{ad}\, F_0 \cdot F_1 = F_{01}$.
(A3) Non-exceptional case: $H_s(z(t))$ is nonzero.

Then, we have the following result from Bonnard and Kupka (1993): let t_c be the first conjugate time. Under assumptions (A1)-(A2)-(A3), the reference singular trajectory $x(\cdot)$ is $\mathcal{C}^0$-locally time minimizing in the hyperbolic case and time maximizing in the elliptic case on $[0\,,t_c)$. Moreover, $x(\cdot)$ is not time optimal on $[0\,,t]$ in L^∞ topology for every $t > t_c$.

3. APPLICATION TO THE REDUCED MODEL

3.1 The reduced model when $\varepsilon = 0$

Putting $\varepsilon = 0$ in equation (4'), we get

$$0 = \frac{1}{2}\frac{\rho(h)\, S\, v}{m}\, u - \frac{g}{v}\cos\gamma. \qquad (12)$$

Solving (12) considering γ is small gives

$$\bar{u} = \frac{2\, m\, g}{\rho(h)\, S\, v^2}.$$

Plugging $\bar{u}$ in equations (1)-(2)-(3) and taking γ as the new control variable gives the following system:

$$\dot{x} = F_0(x) + u\, F_1(x),$$

with $x := (h, v, m)$, $u := \gamma$,

$$F_0(x) = \left(\frac{T(h)}{m} - \frac{1}{2}\frac{\rho(h)\, S\, v^2}{m} C_{D_1} - \frac{2\, m\, g^2}{\rho(h)\, S\, v^2} C_{D_2} \right) \frac{\partial}{\partial v} - (C_s(v)\, T(h)) \frac{\partial}{\partial m},$$

$$F_1(x) = v \frac{\partial}{\partial h} - g \frac{\partial}{\partial v}.$$

The optimal control problem is then:

$$(P_0) \begin{cases} \min t_f, \\ \dot{x}(t) = F_0(x(t)) + u(t)\, F_1(x(t)), \\ \qquad u_{\min} \leq u(t) \leq u_{\max}, \quad t \in [0, t_f] \ a.e., \\ x(0) = x_0, \\ x(t_f) = x_f. \end{cases}$$

We consider a realistic case where the initial state is $x_0 := (3480, 151.67, 69000)$ [5], the final state is $x_f := (9144, 191, 68100)$. The bounds in the control are given by the maximal authorized air slope, $u_{\max} := 0.262$ radian, and we complete symmetrically with $u_{\min} := -u_{\max}$. See table 1 for the chosen values of the constant parameters.

Table 1. Medium-haul aircraft parameters.

Data	Value	Unit	Data	Value	Unit
S	122.6	m^2	C_{s_1}	$1.055e^{-5}$	$kg.s^{-1}.N^{-1}$
g	9.81	$m.s^{-2}$	C_{s_2}	441.54	$m.s^{-1}$
C_{T_1}	141040	N	R	287.058	$J.kg^{-1}.K^{-1}$
C_{T_2}	14909.9	m	Θ_0	288.15	K
C_{T_3}	$6.997e^{-10}$	m^{-2}	β	0.0065	$K.m^{-1}$
C_{D_1}	0.0242		P_0	101325	Pa
C_{D_2}	0.0469				

3.2 Singular extremals

Introducing

$$D_0(x) := \det(F_1(x), F_{01}(x), F_0(x)),$$
$$D_{001}(x) := \det(F_1(x), F_{01}(x), F_{001}(x)),$$
$$D_{101}(x) := \det(F_1(x), F_{01}(x), F_{101}(x)),$$

the singular control is given in feedback form by

$$u_s(x) = -\frac{D_{001}(x)}{D_{101}(x)}, \tag{13}$$

whenever $D_{101}(x) \neq 0$, since, along a singular extremal

$$\langle p, F_1(x)\rangle = \langle p, F_{01}(x)\rangle = \langle p, F_{001}(x) + u\, F_{101}(x)\rangle = 0$$

and $p(\cdot)$ never vanishes. Assuming $D_0(x) \neq 0$, we can write

$$F_{001}(x) - F_{101}(x) = \alpha\, F_0(x) + \alpha^1\, F_1(x) + \alpha^{01}\, F_{01}(x),$$
$$F_{001}(x) + F_{101}(x) = \beta\, F_0(x) + \beta^1\, F_1(x) + \beta^{01}\, F_{01}(x),$$

which gives two relations at x: $D_{001} - D_{101} = \alpha\, D_0$ and $D_{001} + D_{101} = \beta\, D_0$. Assuming $D_{001} - D_{101} \neq 0$ and $D_{001} + D_{101} \neq 0$, the classification from section 2.6 is given by α and β. For instance, the hyperbolic case is given by $\alpha < 0$ and $\beta > 0$. Besides, the generalized Legendre-Clebsch condition (11) becomes $D_0 D_{101} \geq 0$.

[5] The altitute h is given in meter (m), the true air speed v in meter per second ($m.s^{-1}$) and the mass m in kilogram (kg).

3.3 Shooting function and BC-extremal

We use the `Bocop` software, Bonnans et al. (2012), based on direct methods to determine the structure and to get an initial guess for the shooting method we describe hereinafter. This procedure is justified since the indirect methods are very sensitive with respect to the initial guess. The application of the direct method gives a trajectory of the form $\sigma_-\sigma_s\sigma_+$.

We introduce the true Hamiltonians:

$$h_\pm(x, p) := H_0(x, p) \pm H_1(x, p),$$
$$h_s(x, p) := H_0(x, p) + u_s(x)\, H_1(x, p).$$

We define the shooting function $S\colon \mathbb{R}^{5n+3} \to \mathbb{R}^{5n+3}$ by:

$$S(p_0, t_f, t_1, t_2, z_1, z_2) := \begin{bmatrix} H_1(z_1) \\ H_{01}(z_1) \\ h_+(z(t_f)) + p^0 \\ \pi_x\,(z(t_f)) - x_f \\ z(t_1) - z_1 \\ z(t_2) - z_2 \end{bmatrix}, \tag{14}$$

where:

$$z(t_1) := \exp(t_1 \boldsymbol{h}_-)(x_0, p_0),$$
$$z(t_2) := \exp((t_2 - t_1)\boldsymbol{h_s})(z_1),$$
$$z(t_f) := \exp((t_f - t_2)\boldsymbol{h}_+)(z_2).$$

The two first equations are junction conditions with the singular extremal, then we have the boundary conditions and finally the matching conditions. The shooting method consists in solving $S(y) = 0$, with $y := (p_0, t_f, t_1, t_2, z_1, z_2)$. An important point is that to any zero of S is associated a unique BC-extremal of problem (P_0).

We use the `HamPath` code, Caillau et al. (2011), based on indirect methods to solve the shooting equation (14) and we find a solution $\bar{y}$ satisfying $\|S(\bar{y})\| \approx 6e^{-9}$ and given by [6] $\bar{p}_0 \approx (2.673e^{-2}, 0.448, -0.327)$, $\bar{t}_f \approx 644.2$, $\bar{t}_1 = 17.43$ and $\bar{t}_2 = 628.5$. The control law $u(\cdot) = \gamma(\cdot)$ associated to $\bar{y}$ is given in Fig. 1. We check a posteriori that we are in the hyperbolic case and that the strict generalized Legendre-Clebsch condition is satisfied along the singular arc. Hence, the singular extremal is time minimizing up to the first conjugate time.

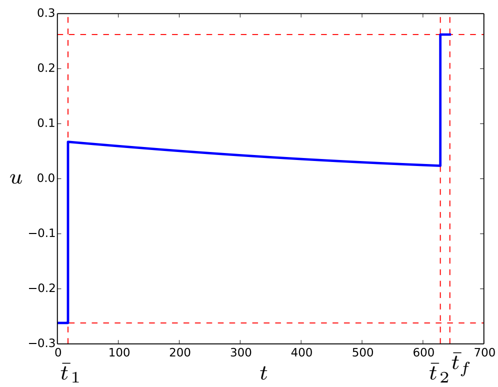

Fig. 1. The control law $u(\cdot) = \gamma(\cdot)$ for the trajectory of the form $\sigma_-\sigma_s\sigma_+$ associated to $\bar{y}$.

[6] We do not give $\bar{z}_1$ nor $\bar{z}_2$ which can be computed by numerical integration.

3.4 Second order conditions of optimality

Since $D_0 D_{101} > 0$ along the singular extremal associated to $\bar{y}$ and because the associated trajectory is injective ($m(\cdot)$ is strictly decreasing), then assumptions (A1), (A2) and (A3) from section 2.7 are satisfied.

To compute the first conjugate time, we can take into account that the dimension of the state is 3. Let $x(\cdot)$ be a reference singular trajectory contained in $D_{101} \neq 0$ defined on $[t_1\,, t_2]$. We define a Jacobi field $J(\cdot)$ along $x(\cdot)$ as a non trivial solution of the variational equation

$$\delta\dot{x}(t) = \frac{\partial}{\partial x}(F_0 + u_s\, F_1)(x(t)) \cdot \delta x(t),$$

satisfying the initial condition $J(t_1) \in \mathbb{R}F_1(x(t_1))$. The first conjugate time is the first time $t_c > t_1$ such that

$$\det\left(J(t_c), F_0(x(t_c)), F_1(x(t_c))\right) = 0.$$

Finally, the singular extremal associated to $\bar{y}$ is time minimizing on $[\bar{t}_1\,, \bar{t}_2]$ according to Fig. 2.

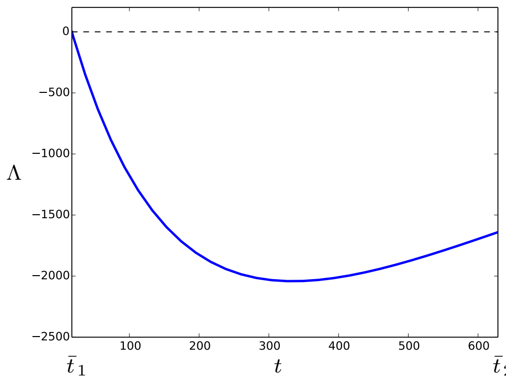

Fig. 2. The determinant $\Lambda := \det(J, F_0, F_1)$, with $J(\bar{t}_1) = F_1(x(\bar{t}_1))$, along the singular arc for the trajectory of the form $\sigma_-\sigma_s\sigma_+$ associated to $\bar{y}$.

4. MODEL WITH SINGULAR PERTURBATION

We are interested in this section in the following optimal control problem:

$$(P_\varepsilon)\begin{cases} \min\, t_f, \\ \dot{x}(t) = F_\varepsilon(x(t), u(t)), \quad u(t) \in \mathbb{R}, \quad t \in [0\,, t_f] \; a.e., \\ x(0) = x_0, \quad x(t_f) = x_f, \end{cases}$$

where $x := (h, v, m, \gamma)$, u is the lift coefficient, the initial state is $x_0 := (3480, 151.67, 69000, 0)$, the final state is $x_f := (9144, 191, 68100, 0)$ and where the dynamics F_ε is given by equations (1)-(2)-(3)-(4'), $\varepsilon \geq 1$ fixed. The values of the constant parameters are given by table 1.

4.1 Multiple shooting and homotopy on ε

The application of the PMP gives the maximizing control:

$$u_\varepsilon(x, p) := \frac{p_\gamma}{2\,\varepsilon\, p_v\, v\, C_{D_2}}$$

and plugging u_ε in $H(x, p, u) := \langle p, F_\varepsilon(x, u)\rangle$ gives the true Hamiltonian $h_\varepsilon(x, p) := \langle p, F_\varepsilon(x, u_\varepsilon(x, p))\rangle$. We use multiple shooting to deal with numerical instability due to the singular perturbation. We note (t_i, z_i) the intermediate discretized times and points. The times t_i are fixed and defined by $t_i = t_{i-1} + \Delta t$, $i = 1, \cdots, k-1$ with $k \in \mathbb{N}^*$, $\Delta t = (t_f - t_0)/k$ and $t_0 := 0$. Then, the multiple shooting function $S^k_\varepsilon(p_0, t_f, z_1, \cdots, z_{k-1})$ is given by:

$$S^k_\varepsilon(p_0, t_f, z_1, \cdots, z_{k-1}) := \begin{bmatrix} \pi_x(z(t_f, t_{k-1}, z_{k-1})) - x_f \\ h_\varepsilon(z(t_f, t_{k-1}, z_{k-1})) + p^0 \\ z(t_1, t_0, z_0) - z_1 \\ \vdots \\ z(t_{k-1}, t_{k-2}, z_{k-2}) - z_{k-1} \end{bmatrix},$$

where $z_0 := (x_0, p_0)$, $z\colon (t_1, t_0, z_0) \mapsto \exp((t_1 - t_0)\boldsymbol{h_\varepsilon})(z_0)$.

We first use multiple shooting method to solve the equation $S^5_{100}(y) = 0$, $y := (p_0, t_f, z_1, \cdots, z_4)$. Let denote by $\bar{y}^5_{100}$ the solution we find. Then, we define the homotopic function $\varphi(y, \varepsilon) := S^5_\varepsilon(y)$. We use the differential path following method from `HamPath` code to solve the equation $\varphi(y, \varepsilon) = 0$ starting from the initial point $(\bar{y}^5_{100}, 100)$ to the target $\varepsilon = 1$. The solution from the homotopy at $\varepsilon = 1$ is used to initialize the multiple shooting method and solve the equation $S^5_1(y) = 0$. Let denote by $\bar{y}^5_1$ this solution. Then, we get $\|S^5_1(\bar{y}^5_1)\| \approx 5e^{-9}$.

We compare the state variable γ when $\varepsilon \in \{1, 10\}$ in Fig. 3 to emphasize the impact of the singular perturbation.

We can see in Fig. 4 that the slow variable h is very well approximated by the one from the reduced-order problem (P_0). In Fig. 5, we see that what we call the outer solution of the fast variable in the singular perturbation theory, is very well approximated by the singular optimal control of problem (P_0). Besides, we get a very small relative gap, around 0.14%, between the final times associated to both solutions of problems (P_0) and (P_1).

4.2 Second order conditions of optimality

We check now the second order conditions of optimality along the path of zeros. For a fixed $\varepsilon \in [1\,, 100]$ we denote by p^ε_0 the initial costate solution of the associated shooting equation. Then, we compute the three Jacobi fields $J_i(\cdot) := (\delta x_i(\cdot), \delta p_i(\cdot))$ such that $\delta x_i(0) = 0$ and where $(\delta p_1(0), \delta p_2(0), \delta p_3(0))$ is a basis satisfying

$$\mathrm{d}h_\varepsilon(x_0, p^\varepsilon_0) \cdot \delta p_i(0) = 0.$$

The first conjugate time $t_c > 0$ is the first time such that $\det(\delta x_1, \delta x_2, \delta x_3, F_\varepsilon)$ vanishes. The determinant being very big when ε is close to 1, we prefer to use a SVD decomposition and compute the smallest singular value. We denote by $\sigma^\varepsilon_{\min}(t)$ the minimal singular value at time t and $\sigma^\varepsilon_{\max}(t)$ the maximal one. Then, $t_c > 0$ is a conjugate time if and only if $\sigma^\varepsilon_{\min}(t_c) = 0$. According to Fig. 6 the BC-extremal along the path of zeros are locally optimal for $\varepsilon \in [1.5\,, 100]$ [7]. When $\varepsilon = 1$, we cannot conclude about the local optimality because of significant round-off errors, see Fig. 7.

5. CONCLUSION

Combining theoretical and numerical tools from geometric control, we get a $\sigma_-\sigma_s\sigma_+$ hyperbolic trajectory, solution of the PMP for the reduced-order problem (P_0). Despite the singular perturbation, we also present BC-extremals

[7] We only represent $\sigma^\varepsilon_{\min}(\cdot)$ for $\varepsilon \in \{1.5, 2, 5, 10\}$ for readibility.

satisfying second order conditions of optimality for problems (P_ε), $\varepsilon > 0$. In conclusion, the numerical comparison between problems (P_0) and (P_1) shows that the reduced-order problem is a very good approximation of the original time-to-climb problem.

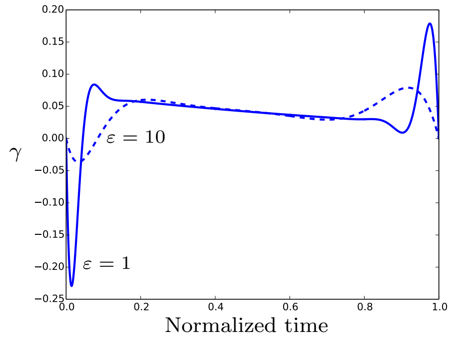

Fig. 3. The air slope $\gamma(\cdot)$ for $\varepsilon = 10$ (dashed line) and $\varepsilon = 1$ (solid line).

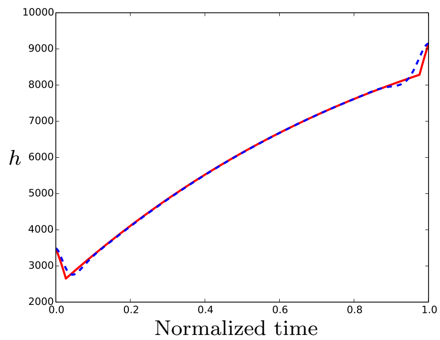

Fig. 4. The altitude $h(\cdot)$ for $\varepsilon = 1$ (blue dashed line) and $\varepsilon = 0$ (red solid line).

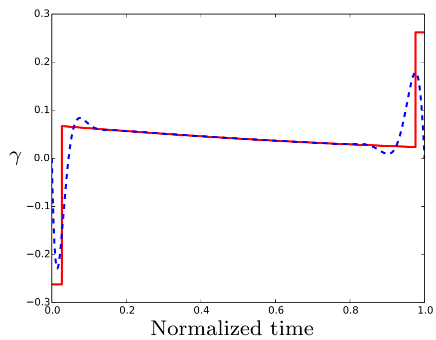

Fig. 5. The air slope $\gamma(\cdot)$ for $\varepsilon = 1$ (blue dashed line) and $\varepsilon = 0$ (red solid line). When $\varepsilon = 0$, $\gamma(\cdot)$ is the control law of problem (P_0).

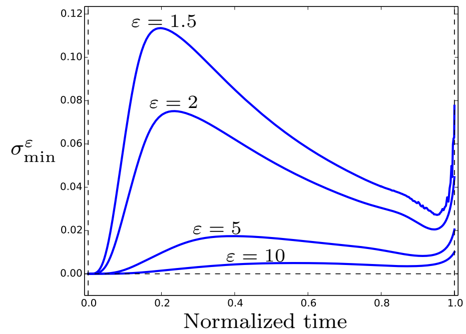

Fig. 6. The singular value $\sigma^\varepsilon_{\min}(\cdot)$ for $\varepsilon \in \{1.5, 2, 5, 10\}$.

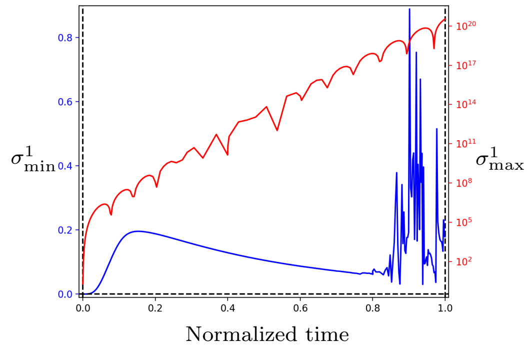

Fig. 7. The singular values $\sigma^\varepsilon_{\min}(\cdot)$ and $\sigma^\varepsilon_{\max}(\cdot)$ for $\varepsilon = 1$.

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