Numerical Analysis in Optimal Control Problem for Aircraft Trajectories

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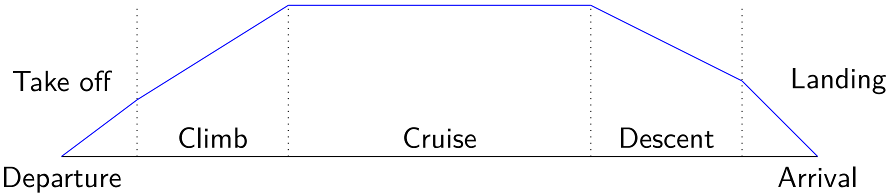
Summary

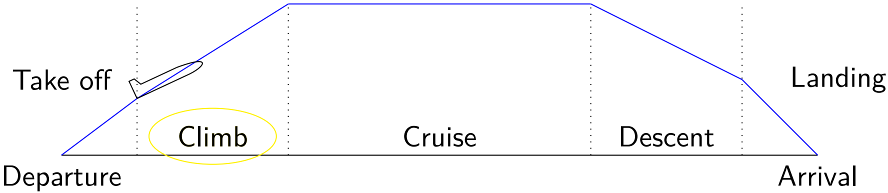
Aircraft trajectories include several flight phases with a lot of constraints



Each phase gets some constraints from :

- Air traffic control
- Flight enveloppe
- Historical practices

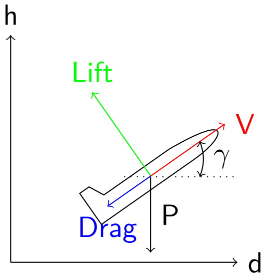
Climb phase for a middle-haul aircraft will be considered in this study



Constraints to be considered for climbing phase :

- maximum speed, or limited climb rate
- climb procedure to follow

Aircraft dynamics :



$$\frac{\mathrm{d}h}{\mathrm{d}t} = V\sin(\gamma)$$
$$\frac{\mathrm{d}d}{\mathrm{d}t} = V\cos(\gamma)$$
$$m\frac{\mathrm{d}V}{\mathrm{d}t} = \varepsilon T_{max}(h) - \overbrace{\frac{1}{2}\rho(h)SV^2C_D(C_L)}^{\text{Drag}} - mg\sin(\gamma)$$
$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\varepsilon C_s(V)T_{max}(h)$$
$$mV\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \underbrace{\frac{1}{2}\rho(h)SV^2C_L}_{\text{Lift}} - mg\cos(\gamma)$$

h : altitude	γ : air slope	C_L,C_D : lift,drag coefficients
d : longitudinal distance	C_s : fuel flow	ε : thrust ratio
V : air speed	T_{max} : maximal Thrust	S : wing area
m : weight	ρ : air density	g : gravitational constant

Atmospheric model :

ISA (*International Standard Atmosphere*) model

$$\begin{cases} T = T_0 - \beta h \\ P = P_0 \left(\frac{T}{T_0}\right)^{\frac{g}{\beta R}} \end{cases} \rightarrow \rho = \frac{P}{RT},$$ R is the specific constant of air.

Aircraft data model :

BADA model

- $T_{max} = C_{T_1}\left(1 - \frac{h}{C_{T_2}} + C_{T_3} h^2\right)$ is the maximal thrust
- $C_s = C_{s_1}\left(1 + \frac{V}{C_{s_2}}\right)$ is the fuel flow
- $C_D = C_{D1} + C_{D2}.C_L^2$ is the drag coefficient

- state : $x = (h, d, v, m, \gamma)$
- control : $u = (\varepsilon, C_L)$
- parameters : $\omega = (S, g, C_{T_1}, C_{T_2}, C_{T_3}, C_{D_1}, C_{D_2}, C_{s_1}, C_{s_2}, R, T_0, \beta, P_0)$
- auxiliary functions : $\theta(x, \omega) = (\theta_1(x, \omega), \theta_2(x, \omega), \theta_3(x, \omega), \theta_4(x, \omega))$

h is the altitude

d is the longitudinal distance

V is the air speed

γ is the air slope

m is the weight

ε is the thrust ratio

C_L is the lift coefficient

Then we write the dynamics :

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,u,\omega,\theta) = f_0(x,\omega,\theta) + u_1 f_1(x,\omega,\theta) + u_2 f_2(x,\omega,\theta) + u_2^2 f_3(x,\omega,\theta)$$

With

$$\left| \begin{array}{l} f_0 = \left(x_3 \sin(x_5), x_3 \cos(x_5), -\omega_6\theta_3 - \omega_2 \sin(x_5), 0, -\frac{\omega_2}{x_3}\cos(x_5)\right)^T \\ f_1 = \left(0,0,\frac{\theta_1}{x_4}, -\theta_1\theta_2, 0\right)^T \\ f_2 = \left(0,0,0,0,\frac{\theta_3}{x_3}\right)^T \\ f_3 = (0,0,-\omega_7\theta_3,0,0)^T \end{array} \right.$$

These considerations lead us to solve an optimal control problem (OCP) written in a *Mayer's formulation* and defined by :

$$
\left|\begin{array}{l}
\min\limits_{(t_f,x,u)} t_f \\
\dfrac{\mathrm{d}x}{\mathrm{d}t}(t) = f(x(t),u(t)),\ \ t \in [0,t_f] \text{ a.e.} \\
u(t) \in \mathcal{U} = \left\{u \in \mathbb{R}^2, u_{1min} \leqslant u_1 \leqslant u_{1max}, u_{2min} \leqslant u_2 \leqslant u_{2max}\right\} \\
x(t) \in \mathcal{X} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^* \times \mathbb{R}^* \times \mathbb{R} \rightarrow \text{no state constraints} \\
x(0) = x_0,\ \ x_f \in \mathcal{X}_f = \{x(t_f) \in \mathcal{X}, b_f(x(t_f)) = 0\} \subset \mathcal{X}
\end{array}\right.
$$

$$
b_f = \begin{pmatrix} x_{1_f} - x_1(t_f) \\ x_{2_f} - x_2(t_f) \\ x_{3_f} - x_3(t_f) \\ x_{5_f} - x_5(t_f) \end{pmatrix},\ \ \text{final weight is free}
$$

$$(OCP)\ :\ \begin{cases} \min\ t_f \\ \frac{\mathrm{d}x}{\mathrm{d}t}(t)=f(x(t),u(t)),\ x(0)-x_0=0,\ b_f(x(t_f))=0,\ u\in\mathcal{U},\ t\in[0,t_f]\ \text{a.e} \end{cases}$$

Hamiltonian : H(p(t),x(t),u(t))= $\langle p(t),f(x(t),u(t))\rangle$

• Necessary conditions (PMP) : If (t_f^*, x^*, u^*) is optimal then $\exists p^* \in AC\left([0,t_f],(\mathbb{R}^n)^*\right)$, $(p^*, p^0) \neq (0,0)$ such as a.e :

$$\left| \begin{array}{r} \dot{x}(t)=\frac{\partial H}{\partial p}(p^*(t),x^*(t),u^*(t)) \\ \dot{p}(t)=-\frac{\partial H}{\partial x}(p^*(t),x^*(t),u^*(t)) \\ H(p^*(t),x^*(t),u^*(t))=\max\limits_{u\in\mathcal{U}} H(p^*(t),x^*(t),u) \end{array} \right.$$

Besides, it exists $\lambda = (\lambda_1, \ldots, \lambda_4) \in \mathbb{R}^4$ such as :

$$\left| \begin{array}{l} p^*(t_f)=\sum_{k=1}^4 \lambda_k \frac{\partial b_{f_k}}{\partial x}(x^*(t_f)) \\ H(p^*(t_f),x^*(t_f),u^*(t_f))=-p^0 \end{array} \right.$$

Hamiltonian : $H = H_0 + u_1 H_1 + u_2 H_2 + u_2^2 H_3$, $H_i = \langle p, f_i \rangle$, for $i = 0, \ldots, 3$

- We define the sets :
  $\Sigma_1 = \{H_1 = 0, \text{ and } H_3 \neq 0\}$
  $\Sigma_2 = \{H_1 = H_2 = H_3 = 0\}$
  $\Sigma_3 = \{H_1 = 0, H_2 \neq 0, H_3 = 0\}$

Extremals outside $\Sigma = \Sigma_1 \bigcup \Sigma_2 \bigcup \Sigma_3$ are controled by $u = (u_1, u_2)$ defined by :

$$\begin{cases} u_1(t) = \begin{cases} u_{1\max}, \text{ if } H_1 > 0 \\ u_{1\min}, \text{ if } H_1 < 0 \end{cases} \\ u_2(t) = -\frac{H_2}{2H_3}, \text{ if } H_3 < 0, \text{ else } \begin{cases} u_{2\max}, \text{ if } H_2 > 0 \\ u_{2\min}, \text{ if } H_2 < 0 \end{cases} \end{cases}$$

- Transervality conditions : $p_4^*(t_f) = 0, \; H(p^*(t_f), x^*(t_f), u^*(t_f)) = -p^0$

Definition : Considering $F_1,\ F_2$ two vector field and $z = (x, p) \in \mathcal{X} \times (\mathbb{R}^n)^*$

Lie Bracket : $F_{12} = [F_1, F_2] = \frac{\partial F_1}{\partial x} F_2(x) - \frac{\partial F_2}{\partial x} F_1(x)$
Poisson Bracket :

$$H_{12} = \{H_1, H_2\}(z) = \mathrm{d}H_1(\overrightarrow{H_2})(z) = H_{[F_1,F_2]}(z),\ \overrightarrow{H_2} = \begin{pmatrix} \nabla_p H_2 \\ -\nabla_q H_2 \end{pmatrix}$$

Lemma : The vector family $(f_0, f_1, f_2, f_3, f_{02})$ constitutes a basis of $\mathbb{R}^5$ for $x \in \bar{\mathcal{X}} = \{x \in \mathbb{R}^5, x_1 \neq \frac{\omega_{11}}{\omega_{12}}, x_1 \neq \frac{1+\sqrt{1-4\omega_4^2\omega_5}}{2\omega_4\omega_5}, x_1 \neq \frac{1-\sqrt{1-4\omega_4^2\omega_5}}{2\omega_4\omega_5}, x_3 \neq 0, x_3 \neq \frac{\omega_8}{\omega_9}, x_4 \neq 0\}$.

Proof idea :

$$\det(f_0, f_1, f_2, f_3, f_{02}) = -\theta_1\theta_2\theta_3$$

Proposition : The singular set $\Sigma_2 = \{H_1 = H_2 = H_3 = 0\}$ cannot contain any extremals satisfying the transversality condtions.

Proof idea :

-Extremals in $\Sigma_2 \implies p \perp (f_0, f_1, f_2, f_3, f_{02}) \implies p = 0$

$$\implies H = p^0 = 0 \implies \text{impossible}$$

Hamiltonian : $H = H_0 + u_1 H_1 + u_2 H_2 + u_2^2 H_3$, $H_i = \langle p, f_i \rangle$, for $i = 0, \ldots, 3$

- Considering the particular case $H_1 = 0, H_2 \neq 0, H_3 \neq 0$ inside the set $\Sigma_1 = \{H_1 = 0, \text{ and } H_3 \neq 0\}$:
  Then

$$\frac{\partial H}{\partial u} = (0, H_2 + u_2 H_3) \Longrightarrow \begin{cases} u_1 = \; ? \\ u_2 = -\dfrac{H_2}{2H_3}, \text{ if } H_3 < 0, \text{ else } \begin{cases} u_{2\max}, \text{ if } H_2 > 0 \\ u_{2\min}, \text{ if } H_2 < 0 \end{cases} \end{cases}$$

Determine u_1 **:**

$\triangleright \dot{H}_1 = 0 = \{H_1, H\} = H_{10} + u_2 H_{12} + u_2^2 H_{13} \rightarrow u_1$ does not appear

$\triangleright \ddot{H}_1 = 0 = u_1(H_{101} + u_2 H_{121} + u_2^2 H_{131}) + H_{100} + u_2(H_{102} + H_{120}) + u_2^2(H_{103} + H_{130} + H_{122}) + u_2^3(H_{123} + H_{132}) + u_2^4 H_{133}$

where $H_{101} = \{\{H_1, H_0\}, H_1\}$

Bocop is a software using Direct Methods to solve Optimal Control Problem. This software transform an OCP into a Non linear Problem which is solved by the well known solver IPOPT.

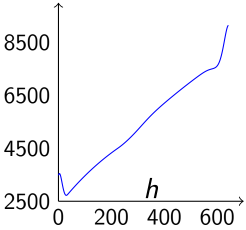
Data :

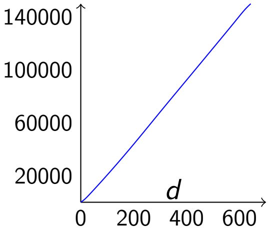
$$\left| \begin{array}{ll} u_{1min} = 0.3, & u_{1max} = 1 \\ u_{2min} = 0, & u_{2max} = 1.6 \end{array} \right.' \quad \left| \begin{array}{l} x_0 = (3480, 0, 145, 64000, 0.25)^T \\ x_f = (9144, 150\ 000, 191, \text{free}, 0)^T \end{array} \right.$$

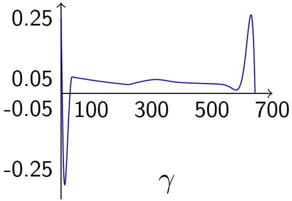
Numerical Test :

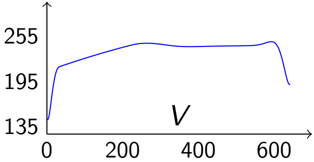
- Obtained with tolerance of 10^{-5}
- Use of Gauss Implicit scheme (second order), with 500 discretised points

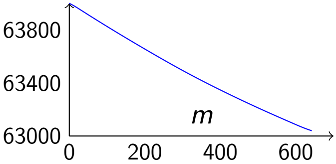
Results from the unconstrained problem :









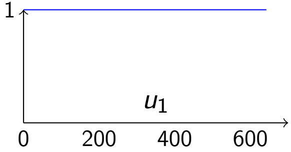


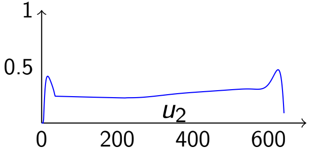
Solution is :

$$t_f = 643s$$

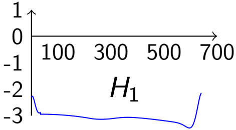
$$x_{4_f} = 63039\text{kg}$$

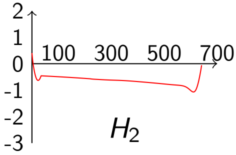
Controls from Bocop are :

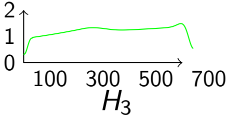




Computed Hamiltonian lifts are :







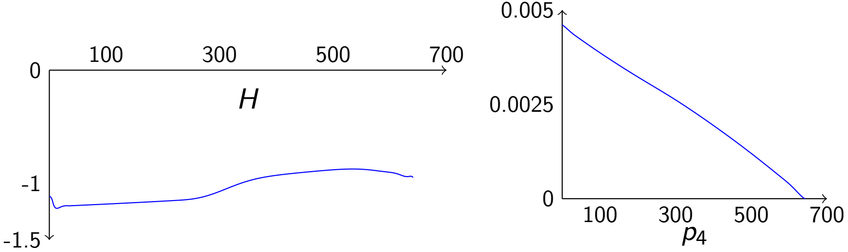
Average error on $|{u_2}_{theo} - {u_2}_{num}| = 1{,}52.10^{-3}$ and $||{u_2}_{theo} - {u_2}_{num}||_\infty = 5{,}8.10^{-2}$

This confirms that control follows the law defined by :

$$u_1(t) = \begin{cases} u_{1\max}, & \text{if } H_1 < 0 \\ u_{1\min}, & \text{if } H_1 > 0 \end{cases}, \quad u_2(t) = -\frac{H_2}{2H_3} \text{ if } H_3 > 0, \text{ else } \begin{cases} u_{2\max}, & \text{if } H_2 < 0 \\ u_{2\min}, & \text{if } H_2 > 0 \end{cases}$$

Transversality conditions give : $H(p^*(t_f), x^*(t_f), u^*(t_f)) = -1, p_4^*(t_f) = 0$
As the system is autonomous, we have : $H(p^*(t), x^*(t), u^*(t)) = \text{Cst} = -1$

The Hamiltonian extracted from Bocop :



Numerical results confirm theoretical behavior of this climbing problem.

This work is the preliminary part of a deeper investigation which includes :

- Numerical solution of this problem using indirect method
- Study of minimum time problem with saturated state constraints, especially on x_5
- Study of a minimal fuel consumption trajectory : $g(t_f, x(t_f)) = -x_4(t_f)$