Numerical Analysis in Optimal Control Problem for Aircraft Trajectories

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British-French-German Conference on Optimisation

15-17 June 2015, London UK
Summary

1. Context of the study
2. Dynamic
3. Pontryagin’s Maximum Principle
4. Numerical Results
5. Future work
Aircraft trajectories include several flight phases with a lot of constraints:

- Take off
- Climb
- Cruise
- Descent
- Landing

Each phase gets some constraints from:
- Air traffic control
- Flight envelope
- Historical practices
Climb phase for a middle-haul aircraft will be considered in this study.

Constraints to be considered for climbing phase:
- maximum speed, or limited climb rate
- climb procedure to follow
Aircraft dynamics:

\[ \frac{dh}{dt} = V \sin(\gamma) \]
\[ \frac{dd}{dt} = V \cos(\gamma) \]
\[ m \frac{dV}{dt} = \varepsilon T_{\text{max}}(h) - \frac{1}{2} \rho(h)SV^2 C_D(C_L) - mg \sin(\gamma) \]
\[ \frac{dm}{dt} = -\varepsilon C_s(V) T_{\text{max}}(h) \]
\[ mV \frac{d\gamma}{dt} = \frac{1}{2} \rho(h)SV^2 C_L - mg \cos(\gamma) \]

- \( h \): altitude
- \( d \): longitudinal distance
- \( V \): air speed
- \( m \): weight
- \( \gamma \): air slope
- \( C_s \): fuel flow
- \( C_L, C_D \): lift, drag coefficients
- \( T_{\text{max}} \): maximal Thrust
- \( \rho \): air density
- \( g \): gravitational constant
- \( S \): wing area
Atmospheric model:

**ISA (International Standard Atmosphere) model**

\[
\begin{align*}
T &= T_0 - \beta h \\
P &= P_0 \left( \frac{T}{T_0} \right) \frac{g}{\beta R} \rightarrow \rho = \frac{P}{R T}, \ R \ is \ the \ specific \ constant \ of \ air.
\end{align*}
\]

Aircraft data model:

**BADA model**

\[
\begin{align*}
T_{\text{max}} &= C_T \left( 1 - \frac{h}{C_T^2} + C_T^3 h^2 \right) \text{ is the maximal thrust} \\
C_s &= C_{s1} \left( 1 + \frac{V}{C_{s2}} \right) \text{ is the fuel flow} \\
C_D &= C_{D1} + C_{D2} \cdot C_L^2 \text{ is the drag coefficient}
\end{align*}
\]
Equation of motion
Normalisation
An OCP

- state: \( x = (h, d, v, m, \gamma) \)
- control: \( u = (\varepsilon, C_L) \)
- parameters: \( \omega = (S, g, C_{T_1}, C_{T_2}, C_{T_3}, C_{D_1}, C_{D_2}, C_{s_1}, C_{s_2}, R, T_0, \beta, P_0) \)
- auxiliary functions: \( \theta(x, \omega) = (\theta_1(x, \omega), \theta_2(x, \omega), \theta_3(x, \omega), \theta_4(x, \omega)) \)

<table>
<thead>
<tr>
<th>h is the altitude</th>
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<tr>
<td>m is the weight</td>
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<tr>
<td>( \varepsilon ) is the thrust ratio</td>
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<tr>
<td>( C_L ) is the lift coefficient</td>
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Then we write the dynamics:

\[
\frac{dx}{dt} = f(x, u, \omega, \theta) = f_0(x, \omega, \theta) + u_1 f_1(x, \omega, \theta) + u_2 f_2(x, \omega, \theta) + u_2^2 f_3(x, \omega, \theta)
\]

With:

\[
f_0 = \begin{pmatrix}
    x_3 \sin(x_5), x_3 \cos(x_5), -\omega_6 \theta_3 - \omega_2 \sin(x_5), 0, -\frac{\omega_2}{x_3} \cos(x_5)
\end{pmatrix}^T
\]

\[
f_1 = \begin{pmatrix}
    0, 0, \frac{\theta_1}{x_4}, -\theta_1 \theta_2, 0
\end{pmatrix}^T
\]

\[
f_2 = \begin{pmatrix}
    0, 0, 0, 0, \theta_3
\end{pmatrix}^T
\]

\[
f_3 = \begin{pmatrix}
    0, 0, -\omega_7 \theta_3, 0, 0
\end{pmatrix}^T
\]
These considerations lead us to solve an optimal control problem (OCP) written in a *Mayer’s formulation* and defined by:

\[
\begin{align*}
\min_{(t_f,x,u)} & \quad t_f \\
\text{subject to} & \quad \frac{dx}{dt}(t) = f(x(t), u(t)), \quad t \in [0, t_f] \text{ a.e.} \\
& \quad u(t) \in \mathcal{U} = \{ u \in \mathbb{R}^2, u_{1\text{min}} \leq u_1 \leq u_{1\text{max}}, u_{2\text{min}} \leq u_2 \leq u_{2\text{max}} \} \\
& \quad x(t) \in \mathcal{X} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^* \times \mathbb{R}^* \times \mathbb{R} \rightarrow \text{no state constraints} \\
& \quad x(0) = x_0, \quad x_f \in \mathcal{X}_f = \{ x(t_f) \in \mathcal{X}, b_f(x(t_f)) = 0 \} \subset \mathcal{X} \\
& \quad b_f = \begin{pmatrix} x_{1f} - x_1(t_f) \\ x_{2f} - x_2(t_f) \\ x_{3f} - x_3(t_f) \\ x_{5f} - x_5(t_f) \end{pmatrix}, \quad \text{final weight is free}
\end{align*}
\]
**General Principle**

**Application to the Climbing Phase**

(OCP) : \[
\begin{align*}
&\min t_f \\
&\frac{dx}{dt}(t)=f(x(t),u(t)), \ x(0)-x_0=0, \ b_f(x(t_f))=0, \ u\in U, \ t\in[0,t_f] \ a.e
\end{align*}
\]

Hamiltonian : \[H(p(t),x(t),u(t)) = \langle p(t),f(x(t),u(t)) \rangle\]

- **Necessary conditions (PMP)**: If \((t^*_f, x^*, u^*)\) is optimal then \[\exists p^* \in \text{AC}([0, t_f], (\mathbb{R}^n)^*), \ (p^*, p^0) \neq (0, 0)\] such as a.e:

\[
\begin{align*}
\dot{x}(t) &= \frac{\partial H}{\partial p}(p^*(t),x^*(t),u^*(t)) \\
\dot{p}(t) &= -\frac{\partial H}{\partial x}(p^*(t),x^*(t),u^*(t)) \\
H(p^*(t),x^*(t),u^*(t)) &= \max_{u\in U} H(p^*(t),x^*(t),u)
\end{align*}
\]

Besides, it exists \(\lambda = (\lambda_1, \ldots, \lambda_4) \in \mathbb{R}^4\) such as:

\[
\begin{align*}
p^*(t_f) &= \sum_{k=1}^{4} \lambda_k \frac{\partial b_{f_k}}{\partial x}(x^*(t_f)) \\
H(p^*(t_f),x^*(t_f),u^*(t_f)) &= -p^0
\end{align*}
\]
Hamiltonian: \[ H = H_0 + u_1 H_1 + u_2 H_2 + u_2^2 H_3 \] \[ H_i = \langle p, f_i \rangle, \text{ for } i=0,\ldots,3 \]

- We define the sets:
  \[ \Sigma_1 = \{ H_1 = 0, \text{ and } H_3 \neq 0 \} \]
  \[ \Sigma_2 = \{ H_1 = H_2 = H_3 = 0 \} \]
  \[ \Sigma_3 = \{ H_1 = 0, H_2 \neq 0, H_3 = 0 \} \]

Extremals outside \( \Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \) are controlled by \( u = (u_1, u_2) \) defined by:

\[
\begin{align*}
  u_1(t) &= \begin{cases} 
    u_{1\text{max}}, & \text{if } H_1 > 0 \\
    u_{1\text{min}}, & \text{if } H_1 < 0 
  \end{cases} \\
  u_2(t) &= -\frac{H_2}{2H_3}, & \text{if } H_3 < 0, \text{ else } \begin{cases} 
    u_{2\text{max}}, & \text{if } H_2 > 0 \\
    u_{2\text{min}}, & \text{if } H_2 < 0 
  \end{cases}
\end{align*}
\]

- Transversality conditions: \( p^*_4(t_f) = 0, \) \( H(p^*(t_f), x^*(t_f), u^*(t_f)) = -p^0 \)
Definition: Considering $F_1$, $F_2$ two vector field and $z = (x, p) \in \mathcal{X} \times (\mathbb{R}^n)^*$

Lie Bracket: $F_{12} = [F_1, F_2] = \frac{\partial F_1}{\partial x} F_2(x) - \frac{\partial F_2}{\partial x} F_1(x)$

Poisson Bracket:
$$H_{12} = \{H_1, H_2\}(z) = dH_1(\vec{H}_2)(z) = H_{[F_1,F_2]}(z), \quad \vec{H}_2 = \left(\begin{array}{c} \nabla_p H_2 \\ -\nabla_q H_2 \end{array}\right)$$

Lemma: The vector family $(f_0, f_1, f_2, f_3, f_{02})$ constitutes a basis of $\mathbb{R}^5$ for

$$x \in \tilde{\mathcal{X}} = \{ x \in \mathbb{R}^5, x_1 \neq \frac{\omega_1}{\omega_2}, x_1 \neq \frac{1 + \sqrt{1 - 4\omega_4^2 \omega_5}}{2 \omega_4 \omega_5}, x_1 \neq \frac{1 - \sqrt{1 - 4\omega_4^2 \omega_5}}{2 \omega_4 \omega_5}, x_3 \neq 0, x_3 \neq \frac{\omega_8}{\omega_9}, x_4 \neq 0 \}.$$

Proof idea:
$$\det (f_0, f_1, f_2, f_3, f_{02}) = -\theta_1 \theta_2 \theta_3$$

Proposition: The singular set $\Sigma_2 = \{H_1 = H_2 = H_3 = 0\}$ cannot contain any extremals satisfying the transversality conditions.

Proof idea:
-Extremals in $\Sigma_2 \implies p \perp (f_0, f_1, f_2, f_3, f_{02}) \implies p = 0$

$$\implies H = p^0 = 0 \implies \text{impossible}$$
Hamiltonian : $H = H_0 + u_1 H_1 + u_2 H_2 + u_2^2 H_3$ , $H_i=\langle p,f_i \rangle$, for $i=0,...,3$

- Considering the particular case $H_1 = 0, H_2 \neq 0, H_3 \neq 0$ inside the set $\Sigma_1 = \{ H_1 = 0, \text{ and } H_3 \neq 0 \}$ :

  Then

  $$\frac{\partial H}{\partial u} = (0, H_2 + u_2 H_3) \implies \begin{cases} u_1 = ? \\ u_2 = -\frac{H_2}{2H_3}, \text{ if } H_3 < 0, \text{ else } \end{cases} \begin{cases} u_{2\text{max}}, \text{ if } H_2 > 0 \\ u_{2\text{min}}, \text{ if } H_2 < 0 \end{cases}$$

**Determine $u_1$** : 

\[ \dot{H}_1 = 0 = \{ H_1, H \} = H_{10} + u_2 H_{12} + u_2^2 H_{13} \rightarrow u_1 \text{ does not appear} \]

\[ \ddot{H}_1 = 0 = u_1(H_{101} + u_2 H_{121} + u_2^2 H_{131}) + H_{100} + u_2(H_{102} + H_{120}) + u_2^2(H_{103} + H_{130} + H_{122}) + u_2^3(H_{123} + H_{132}) + u_2^4 H_{133} \]

where $H_{101} = \{ \{ H_1, H_0 \}, H_1 \}$
Bocop is a software using Direct Methods to solve Optimal Control Problem. This software transform an OCP into a Non linear Problem which is solved by the well known solver IPOPT.

**Data :**

\[
\begin{align*}
  u_{1\text{min}} &= 0.3, & u_{1\text{max}} &= 1 \\
  u_{2\text{min}} &= 0, & u_{2\text{max}} &= 1.6
\end{align*}
\]

\[
\begin{align*}
  x_0 &= (3480, 0, 145, 64000, 0.25)^T \\
  x_f &= (9144, 150 000, 191, \text{free}, 0)^T
\end{align*}
\]

**Numerical Test :**

- Obtained with tolerance of $10^{-5}$
- Use of Gauss Implicit scheme (second order), with 500 discretised points
Results from the unconstrained problem:

Solution is:

\[ t_f = 643s \]
\[ x_{4f} = 63039kg \]
Controls from Bocop are:

\[ u_1(t) = \begin{cases} 
  u_{1\text{max}}, & \text{if } H_1 < 0 \\
  u_{1\text{min}}, & \text{if } H_1 > 0 
\end{cases} \]

\[ u_2(t) = -\frac{H_2}{2H_3}, \text{ if } H_3 > 0, \text{ else } \begin{cases} 
  u_{2\text{max}}, & \text{if } H_2 < 0 \\
  u_{2\text{min}}, & \text{if } H_2 > 0 
\end{cases} \]

Computed Hamiltonian lifts are:

\[ H_1 \]

\[ H_2 \]

\[ H_3 \]

Average error on \(|u_{2\text{theo}} - u_{2\text{num}}| = 1.52 \times 10^{-3}\) and \(\|u_{2\text{theo}} - u_{2\text{num}}\|_\infty = 5.8 \times 10^{-2}\)

This confirms that control follows the law defined by:
Transversality conditions give: \( H(p^*(t_f), x^*(t_f), u^*(t_f)) = -1, p_4^*(t_f) = 0 \)

As the system is autonomous, we have: \( H(p^*(t), x^*(t), u^*(t)) = \text{Cst} = -1 \)

Numerical results confirm theoretical behavior of this climbing problem.
This work is the preliminary part of a deeper investigation which includes:

- Numerical solution of this problem using indirect method
- Study of minimum time problem with saturated state constraints, especially on \( x_5 \)
- Study of a minimal fuel consumption trajectory: \( g(t_f, x(t_f)) = -x_4(t_f) \)