Conjugate loci of Riemannian metrics on 2D manifolds related to the Euler top problem

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Collaborators

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B. Bonnard, O. Cots, J.-B. Pomet, N. Shcherbakova. Riemannian metrics on 2D manifolds related to the Euler-Poinsot rigid body problem (2012, preprint);


Euler’s top problem: classical formulation, optimal control model and Euler’s equations.

Integrability via the Serret-Andoyer trasformation.

The Serret-Andoyer metric on a 2D surface: geodesics, normal form and conjugate locus.

Optimal control of a linear chain of three spin-$\frac{1}{2}$ particles as a sub-Riemannian version of the Euler top problem.

First results on the conjugate locus of the tree spins problem.

Perspectives.
The Euler top problem

**Classical Mechanics viewpoint:** a rigid body, modeled by its *inertia ellipsoid* with principal momenta of inertia $I_1$, $I_2$ and $I_3$, freely rotates about its center of mass, which is fixed.

**Optimal control formulation:** the Euler’s top rotations are the extremals of the optimal control problem on $SO(3)$:

\[
(P) \quad \dot{R}(t) = R(t) \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} = \sum_{i=1}^{3} u_i A_i(R(t)),
\]

\[
\frac{1}{2} \int_{0}^{T} \sum_{i=1}^{3} u_i^2 I_i \rightarrow \min_{u(\cdot)}, \quad T \text{ fixed},
\]

where $R(t) \in SO(3)$ and $A_i \in so(3)$ are the elements of the orthonormal basis

\[
A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

verifying

\[
\]
Euler’s equations and integrability

Assume $R(0)$ and $R(T)$ are fixed. Denote $H_i = p(A_i(R))$ and

$$H_u = \sum_{i=1}^{3} u_i H_i - \nu \sum_{i=1}^{3} I_i u_i^2, \quad \nu \in \{0, 1\}.$$  

- **Pontryagin’s maximal principle** $\implies \nu = 1$, $u_i = H_i I_i^{-1}$ and optimal trajectories are projections of the extremals, i.e., solutions of the Hamiltonian system associated to

$$H_n = \frac{1}{2} \left( \frac{H_1^2}{I_1} + \frac{H_2^2}{I_2} + \frac{H_3^2}{I_3} \right).$$

- Since $\dot{H}_i = \{H_n, H_i\}$, we get Euler’s equations:

$$\frac{dH_1}{dt} = H_2 H_3 \left( \frac{1}{I_3} - \frac{1}{I_2} \right), \quad \frac{dH_2}{dt} = H_1 H_3 \left( \frac{1}{I_1} - \frac{1}{I_3} \right), \quad \frac{dH_3}{dt} = H_1 H_2 \left( \frac{1}{I_2} - \frac{1}{I_1} \right).$$
Euler’s equations

Observations:

- In Classical Mechanics $H = (H_1, H_2, H_3)$ describes the vector of the angular momentum of the body written in the moving frame related to the principal axes of inertia of the body, while $u = (u_1, u_2, u_3)$ is the angular velocity vector.

- Euler’s equations are integrable by quadratures using two first integrals: the Hamiltonian $H_n = h$ and $\|H\|^2 = H_1^2 + H_2^2 + H_3^2$. Solutions to Euler’s equations can be seen as the curves on the energy ellipsoid formed by its intersection with the sphere of the constant angular momentum called polhodes in Classical Mechanics. If $I_3 > I_2 > I_1$, the physically realizable motions take place iff $h \in \left[ \frac{\|H\|^2}{2I_1}, \frac{\|H\|^2}{2I_3} \right]$.

- The full system on $SO(3)$ is integrable by quadratures.
Polhodes on the energy ellipsoid
Observations:

- Remaining equations of motion can be obtained using some suitable coordinates on $SO(3)$, for instance, the Euler angles $\Phi_i, i = 1, 2, 3$ so that
  \[ R = \exp(\Phi_1 A_3) \circ \exp(\Phi_2 A_2) \circ \exp(\Phi_3 A_3). \]

- A complete picture of motion can be seen as the rotation of the energy ellipsoid on a fixed plane (Poinsot model), where the point of contact on the ellipsoid moves along a polhode, while its trace on the plane is called a herpolhode:
Serret-Andoer’s transformation

The Serret-Andoyer coordinates is a set of symplectic coordinates $x, y, z, p_x, p_y, p_z$, defined by

$$ H_1 = \sqrt{p_x^2 - p_y^2} \sin y, \quad H_2 = \sqrt{p_x^2 - p_y^2} \cos y, \quad H_3 = p_y, $$

Then

$$ H_n = H_a = \frac{1}{2} \left( \frac{\sin^2 y}{I_1} + \frac{\cos^2 y}{I_2} \right) (p_x^2 - p_y^2) + \frac{p_y^2}{2I_3}. $$

- $x$ and $z$ are cyclic variables, $p_x$ and $p_z$ are first integrals with $p_x = \|H\|$, and moreover, $z = \text{const}$.
- The dynamics on the $(x, y)$ plane is described by solutions to the system

$$ \frac{dx}{dt} = p_x (A \sin^2 y + B \cos^2 y), \quad \frac{dp_x}{dt} = 0, $$

$$ \frac{dy}{dt} = p_y (C - A \sin^2 y - B \cos^2 y), \quad \frac{dp_y}{dt} = (B - A)(p_x^2 - p_y^2) \sin y \cos y. $$

where

$$ A = \frac{1}{I_1}, \quad B = \frac{1}{I_2}, \quad C = \frac{1}{I_3}. $$
Serret-Andoer’s transformation

Observations:

- The Serret-Andoer transformation is not fiber-preserving.


- If $A < B < C$, then $H_a$ defines a Riemannian metric on a 2D surface:

$$g_a = \frac{2}{w(y)}dx^2 + \frac{2}{2C - w(y)}dy^2,$$

where

$$w(y) = 2(A \sin^2 y + B \cos^2 y).$$
Extremals of Serret-Andoer metric

\( (*) \)

\[
\frac{dx}{dt} = p_x (A \sin^2 y + B \cos^2 y), \quad \frac{dp_x}{dt} = 0,
\]

\[
\frac{dy}{dt} = p_y (C - A \sin^2 y - B \cos^2 y), \quad \frac{dp_y}{dt} = (B - A) (p_x^2 - p_y^2) \sin y \cos y.
\]
Extremals of the Serret-Andoer metric

\[(*) \quad \frac{dx}{dt} = p_x(A \sin^2 y + B \cos^2 y), \quad \frac{dp_x}{dt} = 0, \]
\[\frac{dy}{dt} = p_y(C - A \sin^2 y - B \cos^2 y), \quad \frac{dp_y}{dt} = (B - A)(p_x^2 - p_y^2) \sin y \cos y.\]

- Periodicity: \(H_a(y, p_y) = H_a(y + \pi, p_y);\)
- Symmetry: \(H_a(y, p_y) = H_a(y, -p_y), \quad H_a(y, p_y) = H_a(-y, p_y);\)

The phase portrait on the \((y, p_y)\)-plane is of pendulum type with stable equilibrium at \(y = \frac{\pi}{2} (2h = Ap_x^2)\) and unstable ones at \(y = 0 \text{ mod } \pi (2h = Bp_x^2)\). Thus there are two types of periodic trajectories: oscillations \((Ap_x^2 < 2h < Bp_x^2)\) and rotations \((Bp_x^2 < 2h \leq Cp_x^2)\), and separatrices \((2h = Bp_x^2)\).

\(y(t)\) describes the trajectory on the 0 energy level set of the natural mechanical system

\[
\frac{1}{2} \dot{y}^2 + \left( C - \frac{w(y)}{2} \right) \left( \frac{p_x^2 w(y)}{2} - 2h \right) = 0.
\]
Extremals of the Serret-Andoer metric

\[
\begin{align*}
\frac{dx}{dt} &= p_x(A \sin^2 y + B \cos^2 y), \\
\frac{dp_x}{dt} &= 0, \\
\frac{dy}{dt} &= p_y(C - A \sin^2 y - B \cos^2 y), \\
\frac{dp_y}{dt} &= (B - A)(p_x^2 - p_y^2) \sin y \cos y.
\end{align*}
\]

- **Periodicity:** \( H_a(y, p_y) = H_a(y + \pi, p_y); \)
- **Symmetry:** \( H_a(y, p_y) = H_a(y, -p_y), \quad H_a(y, p_y) = H_a(-y, p_y); \)
- The phase portrait on the \((y, p_y)\)-plane is of pendulum type with stable equilibrium at \( y = \frac{\pi}{2} \) (\( 2h = Ap_x^2 \)) and unstable ones at \( y = 0 \mod \pi \) (\( 2h = Bp_x^2 \)). Thus there are two types of periodic trajectories: oscillations (\( Ap_x^2 < 2h < Bp_x^2 \)) and rotations (\( Bp_x^2 < 2h \leq Cp_x^2 \)), and separatrices (\( 2h = Bp_x^2 \)).
- \( y(t) \) describes the trajectory on the 0 energy level set of the natural mechanical system

\[
\frac{1}{2} \dot{y}^2 + \left( C - \frac{w(y)}{2} \right) \left( \frac{p_x^2 w(y)}{2} - 2h \right) = 0.
\]
Proposition. Set $\xi_1 = \frac{2h-AP_x^2}{p_x^2(B-A)}$, $\xi_3 = \frac{C-A}{B-A}$. Then trajectories starting at the point $(x_0, y_0)$ can be parametrized as follows:

i). oscillating trajectories:
\[
\cos^2 y(t) = \frac{\xi_1 \text{cn}^2(Mt + \psi_0|m)}{1 - \xi_1 \text{sn}^2(Mt + \psi_0|m)},
\]
\[
x(t) - x_0 = p_x \left[ Bt - \left( B - A \right) \left( 1 - \xi_1 \right) \frac{\Pi(\xi_1 | \text{am}(M\tau + \psi_0|m)|m) \bigg|_{\tau=0}^{\tau=t}}{M} \right],
\]
\[
m = \frac{\xi_1(\xi_3 - 1)}{\xi_3 - \xi_1}, \quad M = (B - A)p_x \sqrt{\xi_3 - \xi_1}, \quad \text{cn}(\psi_0|m) = \frac{(1 - \xi_1) \cos^2 y_0}{\xi_1 \sin^2 y_0};
\]

ii). rotating trajectories:
\[
\cos^2 y(t) = \frac{\xi_1 \text{cn}^2(Mt + \psi_0|m)}{\xi_1 - \text{sn}^2(Mt + \psi_0|m)},
\]
\[
x(t) - x_0 = p_x \left[ (A + (B - A)\xi_1) t - \left( B - A \right) \left( \frac{\xi_1 - 1}{\xi_1} \right) \frac{\Pi\left( \frac{1}{\xi_1} | \text{am}(M\tau + \psi_0|m)|m \right) \bigg|_{\tau=0}^{\tau=t}}{M} \right],
\]
\[
m = \frac{\xi_3 - \xi_1}{\xi_1(\xi_3 - 1)}, \quad M = (B - A)p_x \sqrt{\xi_1(\xi_3 - 1)}, \quad \text{cn}(\psi_0|m) = \frac{(\xi_1 - 1) \cos^2 y_0}{\xi_1 - \cos^2 y_0}.
\]
Solutions of (*) are quasi-periodic trajectories on the plane \((x, y)\) for all 
\[h \in \left[\frac{A^2}{2}, \frac{C^2}{2}\right] \setminus \left\{\frac{B^2}{2}\right\},\]
the y-variable being \(\frac{4K(m)}{M}\)-periodic along oscillating trajectories and \(\frac{2K(m)}{M}\)-periodic along rotations;

Two oscillating trajectories of (*) issued from the same initial point \((x_0, y_0)\) with initial conditions \((p_x, p_y(0))\) and \((p_x, -p_y(0))\) intersect after a half period \(\frac{2K(m)}{M}\) at 
\[y\left(\frac{2K(m)}{M}\right) = \pi - y_0.\]
Extremals of the Serret-Andoer metric

**Definition.** The time $t_*$ is called *conjugate* to $t_0 = 0$ if the differential of the end-point mapping

$$\exp'_{(x_0,y_0)} : (p_x(0), p_y(0)) \mapsto (x(t), y(t)),$$

degenerates at $t = t_*$. The point $(x(t_*), y(t_*))$ is said conjugate to $(x_0, y_0)$. We denote $t_*^1 = \min |t_*|$ the first conjugate (to 0) time.

**Theorem.**
- Conjugate times along oscillating trajectories of (*) are solutions to

$$B\xi_1 Mt - (B\xi_1 + A(1 - \xi_1)) \int_{\psi_0}^{Mt+\psi_0} \frac{du}{\text{sn}^2(u|m)} = 0.$$

Moreover, $\frac{2K(m)}{M} \leq t_*^1 < \frac{3K(m)}{M}$, and $t_*^1 = \frac{2K(m)}{M}$ along trajectories starting with $p_y(0) = 0$.
- Rotating trajectories contain no conjugate points;
- \begin{align*}
  t_*^1 &= \min_{p_x \in \left[\sqrt{\frac{2h}{A}}, \sqrt{\frac{2h}{B}}\right]} \frac{\pi \sqrt{A}}{\sqrt{2h(B-A)(C-A)}}, \\
  \lim_{p_x \to \sqrt{\frac{2h}{B}}} t_*^1 &= +\infty.
\end{align*}
The polar form of Serret-Andoyer metric

Let \( \varphi : \)

\[
\frac{dy}{\sqrt{C - (A \sin^2 y + B \cos^2 y)}} = d\varphi.
\]

**N.B.** Since all phase trajectories pass through the point \( y = \frac{\pi}{2} \), one can set \( y(0) = \frac{\pi}{2} \), which leads to \( \sin y = \cn(\alpha \varphi | k) \), where

\[
k = \frac{(B - A)}{(C - A)}, \quad \alpha = \sqrt{C - A}.
\]

**Proposition.**

- \( g_a \) admits the Darboux normal form \( d\varphi^2 + m(\varphi)d\theta^2 \), where \( \theta \equiv x \) and

\[
m(\varphi) = \frac{2}{w(\varphi)} = (A \cn^2(\alpha \varphi | k) + B \sn^2(\alpha \varphi | k))^{-1} \in [\sqrt{A^{-1}}, \sqrt{B^{-1}}].
\]

- The Gauss curvature of \( g_a \) along arc-length parametrized geodesics (\( h = \frac{1}{2} \)):

\[
G(\varphi) = \frac{(A + B + C)(w(\varphi) - w_-)(w(\varphi) - w_+)}{w(\varphi)^2},
\]

\[
w_{\pm} = \frac{2(AB + AC + BC) \pm \sqrt{(AB + AC + BC)^2 - 3ABC(A + B + C)}}{A + B + C}.
\]
The polar form of Serret-Andoyer meric

\[ G_{\max} = \frac{(B - A)(C - A)}{A}, \quad G_{\min} = -\frac{(A + B + C)(w_+w_-)^2}{4w_-w_+}. \]
The polar form of Serret-Andoyer metric

\[ \varphi = K(k) \quad \text{and} \quad \alpha = \frac{K(k)}{\varphi(0)} \]

\[ \varphi(0) = \theta \quad \text{and} \quad 2\pi \]

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**Definition.** Let \( \gamma(\cdot) \in T^*M \), be an integral curve of the Hamiltonian vector field \( \vec{H} \in Vec(T^*M) \) issued from \( \gamma_0 \), where \( M \) is a smooth manifold, and \( e^{t\vec{H}} \) is the Hamiltonian flow generated in \( T^*M \) by \( \vec{H} \). Then

\[
J(\cdot) \in T_{\gamma(\cdot)}(T^*M) : \quad J(t) = e^{t\vec{H}} J(0), \quad J(t) \neq 0
\]

is called the *Jacobi field* along \( \gamma \).

**2D Darboux-type metrics.** Consider the metric \( g = d\varphi^2 + m(\varphi)d\theta^2 \), \( m(\varphi) > 0 \). Assume that

(A1) \( \varphi = 0 \) is a parallel solution (i.e. \( m'(0) = 0 \));
(A2) \( m(\varphi) = m(-\varphi) \), and \( m''(0) > 0 \).

- On the level set \( h = \frac{1}{2} \) in the neighborhood of \( \varphi = 0 \) there is a one-parametric family of periodic solutions to the equation \( \dot{\varphi}^2 + p_\theta^2 \mu(\varphi) = 1 \), which describes the evolution of \( \varphi \) along *arc-length parametrized* geodesics.
- The Hamiltonian \( H = \frac{1}{2} (p_\varphi^2 + p_\theta^2 \mu(\varphi)) \), \( \mu(\varphi) = 1/m(\varphi) \), and any extremal is a solution to

\[
(\ast\ast) \quad \dot{\varphi} = p_\varphi, \quad \dot{p}_\varphi = -\frac{1}{2}p_\theta^2 \mu'(\varphi), \quad \dot{\theta} = p_\theta \mu(\varphi), \quad \dot{p}_\theta = 0.
\]
Jacobi fields of Darboux-type metrics

Let \( J(t) = (\delta_1(t), \delta_2(t), \delta_3(t), \delta_4(t)) \), with

\[
\delta_1 = \delta p_\varphi, \quad \delta_2 = \delta p_\theta, \quad \delta_3 = \delta \varphi, \quad \delta_4 = \delta \theta.
\]

Then

\[
\dot{\delta}_1 = -p_\theta \mu'(\varphi) \delta_2 - \frac{p_\theta^2 \mu''(\varphi)}{2} \delta_3, \quad \dot{\delta}_2 = 0,
\]

\[
\dot{\delta}_3 = \delta_1, \quad \dot{\delta}_4 = \mu(\varphi) \delta_2 + p_\theta \mu'(\varphi) \delta_3.
\]
Let \( J(t) = (\delta_1(t), \delta_2(t), \delta_3(t), \delta_4(t)) \), with

\[
\begin{align*}
\delta_1 &= \delta p_\varphi, & \delta_2 &= \delta p_\theta, & \delta_3 &= \delta \varphi, & \delta_4 &= \delta \theta.
\end{align*}
\]

Then

\[
\begin{align*}
\dot{\delta}_1 &= -p_\theta \mu'(\varphi) \delta_2 - \frac{p_\theta^2 \mu''(\varphi)}{2} \delta_3, & \dot{\delta}_2 &= 0, \\
\dot{\delta}_3 &= \dot{\delta}_1, & \dot{\delta}_4 &= \mu(\varphi) \delta_2 + p_\theta \mu'(\varphi) \delta_3.
\end{align*}
\]
Let $J(t) = (\delta_1(t), \delta_2(t), \delta_3(t), \delta_4(t))$, with

$$
\delta_1 = \delta p_\varphi, \quad \delta_2 = \delta p_\theta, \quad \delta_3 = \delta \varphi, \quad \delta_4 = \delta \theta.
$$

Then

$$
\begin{align*}
\dot{\delta}_1 &= -p_\theta \mu'(\varphi) \delta_2 - \frac{p_\theta^2 \mu''(\varphi)}{2} \delta_3, \quad \dot{\delta}_2 = 0, \\
\dot{\delta}_3 &= \delta_1, \quad \dot{\delta}_4 = \mu(\varphi) \delta_2 + p_\theta \mu'(\varphi) \delta_3.
\end{align*}
$$

Therefore,

$$
\ddot{\delta}_3(t) + \frac{p_\theta^2 \mu''(\varphi(t))}{2} \dot{\delta}_3(t) = -p_\theta \mu'(\varphi(t)) \delta_2.
$$
**Proposition.** The vector fields $J_1 = \vec{H}$, $J_2 = \partial_\theta$, $J_3 = p\varphi(t)\partial_\varphi + p_\theta \partial_\theta + t\vec{H}$, and $J_4 = (\delta_1(t), \delta_2(t), \delta_3(t), \delta_4(t))$, where

$$w_1(t) = \frac{p_\theta^3 \mu'(\varphi(t))}{2} \Lambda(\varphi(t)) - \frac{p_\theta \mu(\varphi(t))}{p\varphi(t)},$$

$$w_3(t) = -p_\theta p\varphi(t) \Lambda(\varphi(t)), \quad w_4 = p^2_\varphi(t) \Lambda(\varphi(t)),$$

with

$$\Lambda(\varphi) = \int_{\varphi_0}^{\varphi} \frac{\mu(\bar{\varphi})}{p^3_\varphi(\bar{\varphi})} d\bar{\varphi}, \quad p\varphi(\varphi) = \sqrt{2h - p^2_\theta \mu(\varphi)},$$

form a well-defined basis of Jacobi fields along any given solution of (**).

**N.B.** The vectors $J_3(0), J_4(0)$ are *vertical*, i.e., $J_i(0) \in T_{\gamma_0} (T^*_\pi(\gamma_0) M)$, where $\pi : T^* M \to M$, while $J_1(0), J_2(0)$ are *horizontal*. 
Jacobi fields of Darboux-type metrics

**Proposition.** The vector fields \( J_1 = \vec{H}, J_2 = \partial_\theta, J_3 = p_\varphi(t)\partial_\varphi + p_\theta \partial_\theta + t\vec{H}, \) and \( J_4 = (\delta_1(t), \delta_2(t), \delta_3(t), \delta_4(t)) \), where

\[
w_1(t) = \frac{p_\theta^3 \mu'(\varphi(t))}{2} \Lambda(\varphi(t)) - \frac{p_\theta \mu(\varphi(t))}{p_\varphi(t)},
\]

\[
w_3(t) = -p_\theta p_\varphi(t) \Lambda(\varphi(t)), \quad w_4 = p_\varphi^2(t) \Lambda(\varphi(t)),
\]

with

\[
\Lambda(\varphi) = \int_{\varphi_0}^{\varphi} \frac{\mu(\bar{\varphi})}{p_\varphi^3(\bar{\varphi})} d\bar{\varphi}, \quad p_\varphi(\varphi) = \sqrt{2h - p_\theta^2 \mu(\varphi)},
\]

form a well-defined basis of Jacobi fields along any given solution of (\(\ast\ast\)).

**N.B.** The vectors \( J_3(0), J_4(0) \) are *vertical*, i.e., \( J_i(0) \in T_{\gamma_0}(T^*_\pi(\gamma_0)M) \), where \( \pi : T^*M \to M \), while \( J_1(0), J_2(0) \) are *horizontal*. 
Corollary. For a Darboux type metric $g$ the conjugate (to 0) times along any periodic trajectory issued from the point $(\theta_0, \varphi_0)$ with $p_\varphi(0) \neq 0$ are solutions to the equation

$$\Lambda(\varphi(t)) = 0.$$ 

The points $(\theta, \pm \bar{\phi})$, where $\pm \bar{\phi}$ are the extremities of the variation of $\varphi$, are conjugate to each other.
Corollary. For a Darboux type metric $g$ the conjugate (to 0) times along any periodic trajectory issued from the point $(\theta_0, \varphi_0)$ with $p_\varphi(0) \neq 0$ are solutions to the equation

$$\frac{\partial \theta(\varphi, p_\theta)}{\partial p_\theta} = 0.$$ 

The points $(\theta, \pm \bar{\phi})$, where $\pm \bar{\phi}$ are the extremities of the variation of $\varphi$, are conjugate to each other.
Theorem. The first conjugate locus to a point \((0, \varphi_0)\) of the Serret-Andoyer metric \(g_a\) is formed by the conjugate points along oscillating trajectories only. It consists of two components, symmetric with respect to the vertical line \(\theta = 0\). Each component of the locus is formed by two smooth branches, which asymptotically tend to the horizontal lines \(\varphi = \pm \frac{K(k)}{\alpha}\), and form a unique horizontal cusp on the line \(\varphi = -\varphi_0\). Moreover, along arc-length parametrized geodesics \((h = 1/2)\)

\[
\min_{\varphi_0} \theta_*(\varphi_0) = \theta_*(0) = \frac{\pi A \sqrt{A}}{\sqrt{(B - A)(C - A)}}.
\]
The right-hand side \( (p_\theta > 0) \) components of the conjugate locus on the \((x, y)\) plane:

\[
A = 1.5, \quad B = 2, \quad C = 2.8
\]
\[
x_0 = 0, \quad y_0 = \arccos \sqrt{0.1}
\]
The right-hand side \((p_\theta > 0)\) components of the conjugate locus on the \((x, y)\) plane:

\[
A = 1.5, \quad B = 2, \quad C = 2.8 \\
x_0 = 0, \quad y_0 = \frac{\pi}{2} (\varphi_0 = 0)
\]
Another 2D structure on $SO(3)$

Let

$$\dot{R}(t) = R(t) \begin{pmatrix} 0 & -u_3 & 0 \\ u_3 & 0 & -u_1 \\ -0 & u_1 & 0 \end{pmatrix}, \quad R(t) \in SO(3).$$

Denote $r_i = R_{1i}, i = 1, 2, 3$, then

$$\begin{aligned}
(P_0) \quad & \dot{r}_1 = u_3 r_2, \\
& \dot{r}_2 = -u_3 r_1 + u_1 r_3, \\
& \dot{r}_3 = -u_1 r_2.
\end{aligned}$$

Consider the problem of optimal transfer

$$\begin{aligned}
(P_1) \quad & r(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \\
& r(T) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
(P_2) \quad & \int_0^T (I_1 u_1^2(t) + I_3 u_3^2(t)) dt \to \min, \\
& T - \text{fixed}.
\end{aligned}$$

**N.B.** Since $\|r\| = 1$, optimal trajectories of $(P_0) - (P_2)$ are geodesics of an almost-Riemannian metric on $S^2$:

$$g_k = \frac{dr_1^2 + k^2 dr_3^2}{r_2^2}, \quad k^2 = \frac{I_1}{I_3}.$$
Motivating example: fastest transfer in a linear chain of tree weakly coupled spins

**Physical model:** Consider a linear chain of three spin-$\frac{1}{2}$ particles, coupled with unequal Ising couplings. The Hamiltonian has the form

\[ H = 2J_{12}I_{1z}I_{2z} + 2J_{23}I_{2z}I_{3z} + H_u, \]

where

\[ I_{1\alpha} = I_\alpha \otimes I_0 \otimes I_0, \quad I_{2\alpha} = I_0 \otimes I_\alpha \otimes I_0, \quad I_{3\alpha} = I_0 \otimes I_0 \otimes I_\alpha, \]

\[ I_x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_y = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad I_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]

and $J_{ij}$ are scalar couplings between $i$-th and $j$-th particle.

**Problem:** the fastest transfer $I_{1x} \rightarrow 4I_{1z}I_{2z}I_{3z}$ by the following sequence

\[(P_1) \quad I_{1x} \rightarrow 2I_{1y}I_{2z} \rightarrow 2I_{1y}I_{2x} \rightarrow 4I_{1y}I_{2y}I_{3z}.\]
Motivating example: fastest transfer in a linear chain of tree weakly coupled spins

**Optimal Control Problem:** Denote $\langle O \rangle = \text{Tr}(O\rho)$, and set

$$x_1 = \langle I_1 x \rangle, \quad x_2 = \langle 2I_1 y I_2 z \rangle, \quad x_3 = \langle 2I_1 y I_2 x \rangle, \quad x_4 = \langle 4I_1 y I_2 y I_3 z \rangle.$$

Then

$$\begin{align*}
(P_1) & \implies (P_2): \\
\frac{dX}{dt} &= \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & -u & 0 \\
0 & u & 0 & -1/k \\
0 & 0 & 1/k & 0
\end{pmatrix} X, \quad k = \frac{J_{12}}{J_{23}};
\end{align*}$$

$$X(0) = (1, 0, 0, 0)^T, \quad X(T) = (0, 0, 0, 1)^T, \quad T \to \min.$$

H. Yuan, R.Zeier, N.Khaneja, 2008: geodesics of the metric $g_k$ describe solutions to $(P_2)$ for

$$r_1 = x_1, \quad r_2 = \sqrt{x_2^2 + x_3^2}, \quad r_3 = x_4.$$
Analysis of extremals of \((P_1) - (P_3)\)

- In spherical coordinates such that \(r_2 = \cos \varphi, r_1 = \sin \varphi \cos \theta, r_3 = \sin \varphi \sin \theta\), the Hamiltonian associated to \(g_k\) takes the form

\[
H_k = \frac{1}{4k^2 \sin^2 \varphi} \left( p_\varphi^2 \sin^2 \varphi (\sin^2 \theta + k^2 \cos^2 \theta) + p_\theta^2 \cos^2 \varphi (\cos^2 \theta + k^2 \sin^2 \theta) - 2(k^2 - 1)p_\varphi p_\theta \sin \varphi \cos \varphi \sin \theta \cos \theta \right).
\]

- If \(k = 1\) (\(\sim\) equal Ising couplings) we get a Grushin-type metric with \(H = \frac{1}{4} (p_\varphi^2 + p_\theta^2 \cot^2 \varphi)\).

- The family of metrics \(g_k\) have a fixed singularity on the equator \(\varphi = \frac{\pi}{2}\) and a discrete symmetry group defined by

\[
H_k(\varphi, p_\varphi) = H_k(\pi - \varphi, -p_\varphi), \quad H_k(\theta, p_\theta) = H_k(-\theta, -p_\theta).
\]
Analysis of extremals of \((P_1) - (P_3)\)

Consider

\[
M_0 = \{ R \in SO(3) \ : \ R' = (r(0), \cdot, \cdot), \ r(0) = (1, 0, 0) \}, \\
M_1 = \{ R \in SO(3) \ : \ R' = (r(T), \cdot, \cdot), \ r(T) = (0, 0, 1) \}.
\]

**Proposition.** The extremals of the Riemannian metric \(g_k\) on \(S^2\) with boundary conditions \(r(0), r(T)\) are extremals of the sub-Riemannian problem on \(SO(3)\) with \(k^2 = I_1/I_3\), satisfying the boundary conditions \((R(0), \lambda(0)) \in M_0^\perp, (R(T), \lambda(T)) \in M_1^\perp\), where \(\lambda\) denotes the adjoint vector.

**N.B.** Coming back to \(SO(3)\), we get the Hamiltonian

\[
H_k = \frac{1}{4} \left( \frac{H_1^2}{I_1} + \frac{H_3^2}{I_3} \right).
\]

It can be seen as a limit case of Euler’s top Hamiltonian \(H_n\) for \(I_2 \to +\infty\). Setting \(\cos \vartheta = \frac{H_1}{2\sqrt{I_1}}, \ \sin \vartheta = \frac{H_3}{2\sqrt{I_3}}\), Euler’s equations imply the generalized pendulum equation

\[
\frac{d^2 \vartheta}{dt^2} = \frac{\sin 2\vartheta (k^2 - 1)}{2I_1}, \quad k^2 = \frac{I_1}{I_3}.
\]
Conjugate locus structure, case $H = 1$, $\varphi(0) = \frac{\pi}{2}$, $\theta_0 = 0$: the first insight

Dependence of the first conjugate time upon variations of $k \geq 1$ ($p_\theta = 10^{-4}$):

red curve: $\theta(t_1(p_\theta, k))$, blue curve: $\varphi(t_1(p_\theta, k))$,

$k_1 \approx 1.061$, $k_2 \approx 1.250$, $k_2 \approx 1.429$
Conjugate locus structure, case $H_k = 1$, $\varphi(0) = \frac{\pi}{2}$, $\theta_0 = 0$: the first insight

Evolution of the conjugate locus: $1 \leq k < k_1$:

\[
\begin{align*}
\theta &\quad \phi \\
0 &\quad \pi/2 &\quad \pi
\end{align*}
\]

$k = 1$, $k = 1.05$
Conjugate locus structure, case $H_k = 1$, $\varphi(0) = \frac{\pi}{2}$, $\theta_0 = 0$: the first insight

Evolution of the conjugate locus: $k_1 \leq k < k_2$:

\[ k = 1.1, \quad k = 1.2 \]

**N.B.** A closed loop occurs at $\bar{k} \in [k_1, k_2]$: $\varphi(t_1(\bar{k})) = \frac{\pi}{2}$, $\bar{k} \approx 1.155$. 
Evolution of the conjugate locus: $k_2 \leq k < k_3$:

\[ k = 1.3, \quad k = 1.4 \]
Conjugate locus structure, case $H_k = 1$, $\varphi(0) = \frac{\pi}{2}$, $\theta_0 = 0$: the first insight

Examples of different types of trajectories for $1 \leq k < k_3$:

$k \in [1, k_3]$  

$k \in [k_1, k_3]$  

$k \in [k, k_3]$  

$k \in [k_2, k_3]$
Conjugate locus structure, case $H_k = 1$, $\varphi(0) = \frac{\pi}{2}$, $\theta_0 = 0$: the first insight

Evolution of the conjugate locus, case $k \leq 1$:

\[ k = 0.8 \]
Conjugate locus structure, case $H_k = 1$, $\varphi(0) = \frac{\pi}{2}$, $\theta_0 = 0$: the first insight

Evolution of the conjugate locus, case $k \leq 1$:

$k = 0.5$
Conjugate locus structure, case $H_k = 1$, $\varphi(0) = \frac{\pi}{2}$, $\theta_0 = 0$: the first insight

Evolution of the conjugate locus, case $k \leq 1$:

$k = 0.2$
Conjugate locus structure, case $H_k = 1$, $\varphi(0) = \frac{\pi}{2}$, $\theta_0 = 0$: the first insight

Evolution of the conjugate locus, case $k \leq 1$:

$k = 0.1$
Conjugate locus structure, case $H_k = 1$, $\varphi(0) = \frac{\pi}{2}$, $\theta_0 = 0$: the first insight

Evolution of the conjugate locus on the sphere:

magenta curve: $k = 1$, red curves: $k = 0.8$, $k = 1.15 < \bar{k}$
Conjugate locus structure, case $H_k = 1$, $\varphi(0) = \frac{\pi}{2}$, $\theta_0 = 0$: the first insight

Evolution of the conjugate locus on the sphere:

magenta curve: $k = 1$, red curves: $k = 1.2$, $k = 1.25$