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THE SERRET-ANDOYER RIEMANNIAN METRIC AND EULER-POINSOT RIGID BODY MOTION

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Abstract. The Euler-Poinsot rigid body motion is a standard mechanical system and is the model for left-invariant Riemannian metrics on $SO(3)$. In this article, using the Serret-Andoyer variables we parameterize the solutions and compute the Jacobi fields in relation with the conjugate locus evaluation. Moreover the metric can be restricted to a 2D surface and the conjugate points of this metric are evaluated using recent work [4] on surfaces of revolution.

1. **Introduction.** Using geometric optimal control, the Euler-Poinsot rigid body motions are the extremals of the left-invariant Riemannian problem on $SO(3)$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \sum_{i=1,3} u_i R A_i, \qquad \min_{u\in L^2[0,T]} \int_0^T \sum_{i=1,3} I_i u_i^2 \mathrm{d}t,$$

with

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $RA_i, i = 1, 2, 3$ form a basis of left-invariant vector fields on $SO(3)$ and the principal moments of inertia of the body are oriented using $I_1 \geq I_2 \geq I_3 > 0$. Conversely by chosing an appropriate frame every left-invariant metric on $SO(3)$ can be written in this form.

From standard geometric analysis, see [1] the following is well known. The extremal curves are solutions of the left-invariant Hamiltonian

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$$H = \frac{1}{2}\left(\frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}\right)$$

where $M = (M_1, M_2, M_3)$ is the angular momentum of the body measured in a specific moving frame. The motion is Liouville integrable and every trajectory is confined by the first integrals to a two-dimensional torus on which the motion is pseudo-periodic and characterized by two frequencies described using the Euler-Poinsot representation. Moreover the integral curves can be computed using elliptic integrals of the first and third kind, see [11].

A set of symplectic variables was introduced by Serret-Andoyer (see the survey [7]) to reduce the Hamiltonian to the form

$$H\left(g, k, l, G, K, L\right) = \frac{1}{2}\left(\frac{\sin^2 l}{I_1} + \frac{\cos^2 l}{I_2}\right)\left(G^2 - L^2\right) + \frac{L^2}{2I_3},$$

where G, K, L are the dual variables associated to g, k, l. If $I_1 = I_2$ the Hamiltonian depends only on momenta, but in general a further transformation is required to integrate the motion using Hamilton-Jacobi method. Finally such a transformation can be easily related to the standard action-angle variables to represent the rigid body motion [10] (they are crucial in perturbation analysis).

The aim of this article is to study the Euler-Poinsot motion from the geometric optimal control point of view. Indeed such a system is related to the attitude control problem of a satellite [3] (assuming a direct control of the angular velocity by impulse torques) and the limit case $I_1 \to +\infty$ defines a left-invariant SR-metric on $SO(3)$ [15] and is associated to the dynamics of spin systems [16]. In this framework the important and difficult problems are to determine the **conjugate** and **cut loci**: fixing $R_0 \in SO(3)$, the conjugate locus $C(R_0)$ is formed by the extremities of extremal curves where optimality is lost for the C^1-neighboring curves while the cut locus $C_{\mathrm{cut}}(R_0)$ is formed by extremities of extremals where the optimality is lost globally. Besides the interest of such computations in optimal control, the analysis of conjugate loci in Riemannian manifold has a long history in geometry, which goes back to Jacobi's study of the conjugate locus on ellipsoids, see [8] for recent advances. The conjugate locus for the rigid body in the case of two equal moment of inertia was descrided very recently in [2].

The main contribution of this article is to describe the conjugate locus of the Serret-Andoyer metric $g_a = \left(\frac{\mathrm{d}x^2}{z(y)} + \frac{\mathrm{d}y^2}{2/I_3 - z(y)}\right)$,where $z(y) = 2\left(\frac{\sin^2 y}{I_1} + \frac{\cos^2 y}{I_2}\right)$ associated to the restriction of H to the (g, l, G, L)-space and the principal moments of inertia being oriented according to $I_1 > I_2 > I_3$. The analysis relies on recent studies of metrics on surfaces of revolution [13] and refined developments on two-spheres of revolution [4], [14]. The key tool is to reduce the metric to the polar form $g = \mathrm{d}\varphi^2 + m(\varphi)\mathrm{d}\theta^2$ which is used to evaluate the Jacobi fields and the conjugate locus. The interest of interpreting the Serret-Andoyer metric in a geometric framework of surfaces of revolution is to show the analogy with the conjugate locus computations on oblate ellipsoids of revolution. It is a first step toward the computations of the conjugate locus for left-invariant metrics on $SO(3)$ through the Jacobi fields parameterized using the polar normal form.

2. Riemannian metrics on surfaces of revolution.

The objective of this section is to introduce the concepts and to recall the properties of the metrics on surfaces of revolution [13], presenting also the recent developments to compute the conjugate locus on two-spheres of revolution [4], [14].

2.1. **Generalities of Riemannian metrics on surfaces of revolution.** Taking a chart (U, q) the metric can be written in polar coordinates as

$$\mathrm{d}s^2 = \mathrm{d}\varphi^2 + m(\varphi)\mathrm{d}\theta^2$$

One use Hamiltonian formalism on T^*U, $\frac{\partial}{\partial p}$ is the vertical space, $\frac{\partial}{\partial q}$ is the horizontal space and $\alpha = p\mathrm{d}q$ is the (horizontal) Liouville form. The associated Hamiltonian is

$$H = \frac{1}{2}\left(p_\varphi^2 + \frac{p_\theta^2}{m(\varphi)}\right)$$

and we denote $\exp t\overrightarrow{H}$ the one-parameter group. Parameterizing by arc-length amounts to fix the level set to $H = 1/2$. Extremal solutions of $\overrightarrow{H}$ are denoted $\gamma : t \to (q(t, q_0, p_0), p(t, q_0, p_0))$ and fixing q_0 it defines the exponential mapping $\exp_{q_0} : (t, p_0) \to q(t, q_0, p_0) = \Pi(\exp t\overrightarrow{H}(q_0, p_0))$ where $\Pi : (q, p) \to q$ is the standard projection. Extremals are solutions of the equations

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = p_\varphi, \qquad \frac{\mathrm{d}p_\varphi}{\mathrm{d}t} = \frac{1}{2}p_\theta^2\frac{m'(\varphi)}{m^2(\varphi)},$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{p_\theta}{m(\varphi)}, \qquad \frac{\mathrm{d}p_\theta}{\mathrm{d}t} = 0.$$

Definition 2.1. The relation $p_\theta = C^t$ is called **Clairaut** relation (in Hamiltonian form) on surfaces of revolution. We have two types of specific solutions: **meridians** for which $p_\theta = 0$ and $\theta(t) = \theta_0$ and **parallels** for which $\frac{d\varphi}{dt}(0) = p_\varphi(0) = 0$ and $\varphi(t) = \varphi(0)$.

Remark 1. From the extremals equations parallels are solutions of $m'(\varphi) = 0$. Integrating $H = 1/2$ as the mechanical systems $(\frac{\mathrm{d}\varphi}{\mathrm{d}t})^2 + \frac{p_\theta^2}{m(\varphi)} = 1$ and denoting $V(\varphi, p_\theta) = \frac{p_\theta^2}{m(\varphi)}$ the **potential** mapping they correspond to local extrema (p_θ being a fixed parameter).

Assumptions. In the sequel we shall assume the following

- (A1) $\varphi = 0$ is a parallel solution and the corrresponding parallel is called the **equator**.
- (A2) The metric is **reflectionally symmetric** with respect to the equator: $m(-\varphi) = m(\varphi)$.

Parameterizing by arc-length, one gets

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 = 1 - \frac{p_\theta^2}{m(\varphi)},$$

hence $\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \pm 1/g$, where

$$g(\varphi, p_\theta) = \sqrt{\frac{m(\varphi)}{m(\varphi) - p_\theta^2}}.$$

By symmetry, one can assume $\theta(0) = 0$, $p_\theta \in [0, \sqrt{m(\varphi(0)})]$ and $\frac{\mathrm{d}\varphi}{\mathrm{d}t}(0) \geq 0$. Moreover one restricts our analysis to extremals such that $\varphi(0) = 0$. Arc-length parameterized extremal γ is defined by $t \to (\varphi(t, p_\theta), \theta(t, p_\theta))$ and $\frac{\mathrm{d}\varphi}{\mathrm{d}t} > 0$ corresponds to an increasing branch and $\frac{\mathrm{d}\varphi}{\mathrm{d}t} < 0$ a decreasing branch. One has

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \pm\frac{1}{g}, \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{p_\theta}{m(\varphi)}$$

and for an increasing branch one can parameterize θ by φ and we get

$$\frac{\mathrm{d}\theta}{\mathrm{d}\varphi} = \frac{g(\varphi, p_\theta)p_\theta}{m(\varphi)} = f(\varphi, p_\theta),$$

where

$$f(\varphi, p_\theta) = \frac{p_\theta}{\sqrt{m(\varphi)}\sqrt{m(\varphi) - p_\theta^2}}.$$

2.2. **Jacobi equation and conjugate locus.** We denote $e_1 = \frac{\partial}{\partial\varphi}$, $e_2 = \frac{1}{\sqrt{m(\varphi)}}\frac{\partial}{\partial\theta}$ the orthonormal basis. Using Hamiltonian formalism, H defines a quadratic form on the cotangent bundle and differentiating $H = 1/2$ along an extremal parameterized by arc-length one gets the relation $\langle p, \delta p\rangle = 0$.

Definition 2.2. If γ is a reference extremal, Jacobi equation is the variational equation

$$\frac{\mathrm{d}\delta z}{\mathrm{d}t} = \frac{\partial\vec{H}(\gamma(t))}{\partial z}\delta z, \quad z = (q, p)$$

and we denote $J(t)$ a Jacobi field, that is a non trivial solution of Jacobi equation. The time t_c is said to be conjugate to $t = 0$ if there exists a Jacobi field $J(t) = (\delta q(t), \delta p(t))$ such that $\delta q(0) = \delta q(t_c) = 0$.

If γ is parameterized by arc-length two Jacobi fields are crucial in our analysis.

- $J_1(t) = (\delta x, \delta p)$ denotes the Jacobi field vertical at time $t = 0$, that is $\delta q(0) = 0$ and such that $\langle p(0), \delta p(0)\rangle = 0$.
- $J_2(t)$ is the (Poincaré) Jacobi field vertical at time $t = 0$ and generated by the tangent to the curve $\lambda \to (q(0), p(0) + \lambda p(0))$.

The following facts are standard in Riemannian geometry [6] but translated in terms of symplectic geometry. First of all $H(p(0) + \lambda p(0)) = (\lambda + 1)^2 H(p(0))$ and the level set is not preserved. Moreover along an extremal we have the following: $q(t, q_0, \lambda p_0) = q(\lambda t, q_0, p_0)$ and $p(t, q_0, \lambda p_0) = \lambda p(\lambda t, q_0, p_0)$. Hence the Poincaré Jacobi field can be easily computed:

$$\delta q(t) = \frac{\partial q\,(t, q_0, (\lambda + 1)p_0)}{\partial\lambda_{|\lambda=0}} = t\frac{\mathrm{d}q}{\mathrm{d}t}(t, q_0, p_0)$$

$$\delta p(t) = \frac{\partial p\,(t, q_0, (\lambda + 1)p_0)}{\partial\lambda_{|\lambda=0}} = p(t, q_0, p_0) + t\frac{dp}{dt}(t, q_0, p_0).$$

In particular $\alpha(J_2)$ is non zero since $\alpha(\frac{\mathrm{d}q(t)}{\mathrm{d}t}) = p(t)\frac{\mathrm{d}q(t)}{\mathrm{d}t} = 2H = 1$. Secondly along an extremal curve one has $\alpha(\frac{\mathrm{d}q(t)}{\mathrm{d}t}) = p(t)\frac{\mathrm{d}q(t)}{\mathrm{d}t} = 2H(q(t), p(t))$. Hence the one-parameter group $\exp t\vec{H}$ preserves the restriction of α to the level set $H = 1/2$. Since $J_1(0)$ is vertical one has $\alpha(J_1(0)) = \alpha(J_1(t)) = 0$. Therefore we have.

Proposition 1. *Conjugate points are given by the relation* $\mathrm{d}\Pi(J_1(t)) = 0$ *and*

$$\mathrm{d}\Pi(J_1(t)) = \left(\frac{\partial\varphi(t,p_\theta)}{\partial p_\theta}, \frac{\partial\theta(t,p_\theta)}{\partial p_\theta}\right).$$

In particular we have at any time the collinearity condition:

$$p_\varphi \frac{\partial\varphi}{\partial p_\theta} + p_\theta \frac{\partial\theta}{\partial p_\theta} = 0.$$

Next we determine the conjugate locus of a point on the equator. First of all we have corollary 7.2.1 [13].

Lemma 2.3. *Let* $p_\theta \in (0, \sqrt{m(0)})$ *such that* $\frac{\mathrm{d}\varphi}{\mathrm{d}t} > 0$ *on* $(0,t)$. *Then in this interval there exists no conjugate times.*

Let $I = (0, \sqrt{m(0)})$ be a positive interval such that for any $p_\theta \in I$, the derivative of the increasing branch starting at the equator vanishes at time $T/4$ and $\varphi_+ = \varphi(T/4)$. Then the trajectory $t \to \varphi(t,p_\theta)$ is periodic with period T given by

$$\frac{T}{4} = \int_0^{\varphi_+} g(\varphi,p_\theta)\mathrm{d}\varphi$$

and the first return to the equator is at time $T/2$ and the variation of θ is given by

$$\Delta\theta = 2\int_0^{\varphi_+} f(\varphi,p_\theta)\mathrm{d}\varphi.$$

Definition 2.4. The mapping $p_\theta \in I \to T(p_\theta)$ is called the **period mapping** and $R : p_\theta \to \Delta\theta$ is called the **first return mapping**.

Definition 2.5. The extremal flow is called **tame** on I if the first return mapping R is such that $R' < 0$.

Proposition 2. *For extremal curves with* $p_\theta \in I$, *in the tame case there exists no conjugate times on* $(0, T/2)$.

Proof. As in [4] if $R' < 0$, the extremal curves initiating from the equator with $p_\theta \epsilon I$ are not intersecting before returning to the equator. As conjugate points are limits of intersecting extremal curves, conjugate points are not allowed before returning to the equator. □

Assumptions. In the tame case we assume the following

(A3) At the equator the Gauss curvature $G = -\frac{1}{\sqrt{m(\varphi)}}\frac{\partial^2\sqrt{m(\varphi)}}{\partial\varphi^2}$ is positive and maximum.

Lemma 2.6. *Under assumption (A3), the first conjugate point along the equator is at time* $\pi/\sqrt{G(0)}$ *and realizes the minimum distance to the cut locus* $C_{\mathrm{cut}}(0)$. *It is a cusp point of the conjugate locus.*

Proof. See for instance [12]. □

Parameterization of the conjugate locus under assumptions (A1-2-3) for $p_\theta \in I$ *and* $\varphi(0) = 0$. The conjugate locus will be computed by continuation, starting from the cusp point at the equator. Let $p_\theta \in I$ and $t \in (T/2, T/2 + T/4)$. One has the formulae

$$\theta(t,p_\theta) = \Delta\theta(p_\theta) + \int_{T/2}^{t} \frac{p_\theta}{m(\varphi)}\mathrm{d}t$$

and on $[T/2, t]$, $\frac{\mathrm{d}\varphi}{\mathrm{d}t} < 0$, $\varphi < 0$. Hence

$$\int_{T/2}^{t} \frac{p_\theta}{m(\varphi)}\mathrm{d}t = \int_{\varphi(t,p_\theta)}^{0} f(\varphi, p_\theta)\mathrm{d}\varphi.$$

According to proposition 7.2.1 [13] conjugate times are given by the relation

$$\frac{\partial \varphi(t, p_\theta)}{\partial p_\theta} = \frac{\partial \theta(t, p_\theta)}{\partial p_\theta} = 0.$$

Hence we deduce the following.

Lemma 2.7. *For $p_\theta \in I$ and conjugate times between $(T/2, T/2 + T/4)$ the conjugate locus is solution of*

$$\frac{\partial \theta(\varphi, p_\theta)}{\partial p_\theta} = 0, \tag{1}$$

where $\theta(\varphi, p_\theta) = \Delta\theta(p_\theta) + \int_\varphi^0 f(\varphi, p_\theta)\mathrm{d}\varphi$.

Analysis of equation (1). One notes $p_\theta \to \varphi_{1c}(p_\theta)$ the solution of the equation initiating from the equator. Differentiating one has

$$\Delta\theta' + \int_\varphi^0 \frac{\partial f}{\partial p_\theta}\mathrm{d}\varphi = 0$$

at $\varphi_{1c}(p_\theta)$. Differentiating again we obtain

$$\Delta\theta'' + \int_{\varphi_{1c}}^0 \frac{\partial^2 f}{\partial p_\theta^2}\mathrm{d}\varphi - \frac{\partial \varphi_{1c}}{\partial p_\theta} \cdot \frac{\partial f}{\partial p_\theta} = 0.$$

One can easily check that $\frac{\partial f}{\partial p_\theta} > 0$ and $\frac{\partial^2 f}{\partial p_\theta^2} > 0$. In particular

$$\frac{\partial \varphi_{1c}}{\partial p_\theta} = \left(\Delta\theta'' + \int_{\varphi_{1c}}^0 \frac{\partial^2 f}{\partial p_\theta^2}\mathrm{d}\varphi\right)\left(\frac{\partial f}{\partial p_\theta}\right)^{-1}.$$

Proposition 3. *If $\Delta\theta'' > 0$ on I, then $\frac{\partial \varphi_{1c}}{\partial p_\theta} \neq 0$ and the curve $p_\theta \to (\varphi_{1c}(p_\theta), \theta_{1c}(p_\theta))$ is a curve defined for $p_\theta \in I$ and with no loop in the plane (φ, θ). In particular it is without cusp point.*

Proof. By definition, the conjugate locus for $p_\theta \in I$ is the envelop of the extremal curves initiating from the equator. One extremity is the cusp point at the equator and the curve can be continuated since it is geometrically clear that the conjugate point is located on $(T/2, T/2 + T/4)$, that is before the second zero of $\frac{\mathrm{d}\varphi}{\mathrm{d}t}$. □

To simplify the computations one use [4].

Lemma 2.8. *We have:*

$$R'(p_\theta) = \frac{T'(p_\theta)}{2p_\theta}.$$

Computations on the ellipsoid of revolution: the oblate case. The ellipsoid of revolution with respect to the z-axis is generated by the curve:

$$y = \sin\varphi,\ z = \varepsilon \cos\varphi$$

where $0 < \varepsilon < 1$ corresponds to the oblate case while $\varepsilon > 1$ is the prolate case. The restriction of the Euclidian metric is

$$g = F_1(\varphi)\mathrm{d}\varphi^2 + F_2(\varphi)\mathrm{d}\theta^2$$

where $F_1 = \cos^2\varphi + \varepsilon^2 \sin^2\varphi$, $F_2 = \sin^2\varphi$.

The case $\varepsilon = 1$ is the round sphere and we shall restrict to the oblate case. In this case the Gauss curvature is positive and increasing from the north pole to the equator.

The metric can be written in the polar form thanks to:

$$\mathrm{d}\Phi = F_1^{1/2}(\varphi)\mathrm{d}\varphi$$

which leads to introduce the elliptic function of the second kind: $\Phi = E(\varphi, k)$ where the modulus is $k^2 = 1 - \varepsilon^2$.

We shall compute the period mapping in the (ψ, θ)-coordinates, $\psi = \pi/2 - \varphi$ and $\psi = 0$ is the equator. The Hamiltonian is

$$H = \frac{1}{2}\left(\frac{p_\varphi^2}{F_1(\varphi)} + \frac{p_\theta^2}{F_2(\varphi)}\right)$$

and with $H = 1/2$, one gets

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{(\cos^2\psi - p_\theta^2)^{1/2}}{\cos\psi(\sin^2\psi + \varepsilon^2\cos^2\psi)^{1/2}}.$$

Denoting $1 - p_\theta^2 = \sin^2\psi_1$ and making the rescaling $Y = \sin\psi_1 Z$, where $Y = \sin\psi$, the equation

$$\frac{(Y^2 + \varepsilon^2(1 - Y^2))^{1/2}}{\left(\sin^2\psi_1\left(1 - \frac{Y^2}{\sin^2\psi_1}\right)\right)^{1/2}}\mathrm{d}Y = \mathrm{d}t$$

becomes

$$\frac{(\varepsilon^2 + Z^2\sin^2\psi_1^2(1 - \varepsilon^2))^{1/2}}{(1 - Z^2)^{1/2}}\mathrm{d}Z = \mathrm{d}t.$$

Hence the formulae for the period mapping is

$$\frac{T}{4} = \int_0^1 \frac{(\varepsilon^2 + Z^2\sin^2\psi_1(1 - \varepsilon^2))^{1/2}}{(1 - Z^2)^{1/2}}\mathrm{d}Z.$$

We introduce:

$$\begin{aligned} \alpha &= \sqrt{\varepsilon^2 + \sin^2\psi_1(1 - \varepsilon^2)}, \\ m^2 &= \frac{\sin^2\psi_1(1 - \varepsilon^2)}{\alpha^2}, \\ m'^2 &= \frac{\varepsilon^2}{\alpha^2}. \end{aligned}$$

Hence we have:

$$\begin{aligned} T &= 4\int_0^1 \frac{\alpha(m'^2 + m^2Z^2)}{\sqrt{(1 - Z^2)(m'^2 + m^2Z^2)}}\mathrm{d}Z \\ &= 4\alpha\left[m'^2K(m) + m^2\int_0^1 \frac{Z^2}{\sqrt{(1 - Z^2)(m'^2 + m^2Z^2)}}\mathrm{d}Z\right] \\ &= 4\alpha m'^2K(m) + 4\alpha m^2\int_0^{K(m)} \mathrm{cn}^2\, u\mathrm{d}u, \end{aligned}$$

where $Z = \mathrm{cn}\, u$. Using [11] one gets:

$$\begin{aligned} T &= 4\alpha(m'^2K(m) + (E(m) - m'^2K(m)) \\ &= 4\alpha E(m). \end{aligned}$$

A straightforward computation gives us

$$T'(p_\theta) < 0 < T''(p_\theta).$$

Hence the conjugate locus can be continued from the horizontal cusp point at the equator to the vertical cusp point at the meridian $\theta = \pi$, when $p_\theta \to 0$. (Observe $\lim_{p_\theta \to 0} R(p_\theta) = \pi$). By symmetry one gets easily the standard astroidal shape of the conjugate locus.

Observe the difference with the prolate case.

In this case the curvature is minimum at the equator, the first return mapping is increasing and the miminum is at $p_\theta \to 0$ and is equal to π. In this case the conjugate locus is constructed by continuation from a vertical cusp at $\theta = \pi$ to an horizontal cusp at the equator.

In both cases the cut loci can be determined from the conjugate locus and the symmetries of the extremal flow: a segment of the equator in the oblate case versus a segment of the meridian in the prolate case.

Remark 2. In the oblate case, one can similarly evaluate the different conjugate loci corresponding to i^{th}-conjugate points, $i = 1, 2, \ldots$. For instance for the second locus, we replace in lemma 2.7, θ by the formulae

$$\theta(\varphi, p_\theta) = 2\Delta\theta(p_\theta) + \int_0^\varphi f(\varphi, p_\theta)\mathrm{d}\varphi,$$

for $t \in [T, T + T/4]$, $\varphi > 0$ and the "four cusps Jacobi conjecture of the conjugate loci" can be checked as an exercice left to the reader.

3. **Left-invariant metrics on $SO(3)$ and the Serret-Andoyer formalism in Euler-Poinsot rigid body motion.**

3.1. **Left-invariant metrics on $SO(3)$.**

We recall the geometric framework using [9], [1]. We note (e_1, e_2, e_3) the fixed frame and $(E_1(t), E_2(t), E_3(t))$ the moving frame attached to the body and formed by principal axis. The position of the body is represented by the matrix $R(t) = (E_1(t), E_2(t), E_3(t))$ and is solution of $\frac{\mathrm{d}R}{\mathrm{d}t} = \sum_{i=1,3} u_i R A_i$. The motion is obtained by minimizing $\int_0^T L\mathrm{d}t$, with $L = \sum_{i=1,3} I_i u_i^2$ where the principal moment of inertia in the distinct case are oriented using $I_1 > I_2 > I_3$. The rigid body dynamics can be derived using Pontryagin maximum principle and appropriate coordinates. Le ξ be an element of $T_R^*(SO(3))$ and denotes $H_i = \xi(RA_i), i = 1, 2, 3$ the symplectic lifts of the vector fields RA_i. The pseudo-Hamiltonian takes the form

$$H = \sum_{i=1,3} u_i H_i - \frac{1}{2} \sum_{i=1,3} I_i u_i^2$$

and the maximization condition of the maximum principle implies $\frac{\partial H}{\partial u} = 0$. Hence $u_i = H_i/I_i$ and plugging such u_i into H we get the true Hamiltonian

$$H_e = \frac{1}{2}\left(\frac{H_1^2}{I_1} + \frac{H_2^2}{I_2} + \frac{H_3^2}{I_3}\right). \tag{2}$$

Note that the vector $\mathbf{H}$=(H_1, H_2, H_3) represents the angular momentum of the body measured in the moving frame and related by $H_i = I_i\Omega_i$ to the angular velocity $\mathbf{\Omega} = (\Omega_1\Omega_2, \Omega_3)$.

The **Euler equation** describing the evolution of the angular velocity is

$$\frac{\mathrm{d}H_i}{\mathrm{d}t} = \mathrm{d}H_i(\overrightarrow{H_e}) = \{H_i, H_e\} \tag{3}$$

and $\{,\}$ denotes the Poisson bracket. The Lie bracket of two matrices is computed using the convention $[A, B] = AB - BA$ and we have the relation $\{H_i, H_j\} = \xi([A_i, A_j])$. Computing

$$[A_1, A_2] = -A_3, [A_1, A_3] = A_2, [A_2, A_3] = -A_1$$

and Euler equation is

$$\frac{\mathrm{d}H_1}{\mathrm{d}t} = H_2 H_3 \left(\frac{1}{I_3} - \frac{1}{I_2}\right)$$
$$\frac{\mathrm{d}H_2}{\mathrm{d}t} = H_1 H_3 \left(\frac{1}{I_1} - \frac{1}{I_3}\right)$$
$$\frac{\mathrm{d}H_3}{\mathrm{d}t} = H_1 H_2 \left(\frac{1}{I_2} - \frac{1}{I_1}\right).$$

The following proposition is standard.

Proposition 4. *The Euler equation is integrable by quadratures using the two first integrals: the Hamiltonian H_e and the Casimir $|H|^2 = H_1^2 + H_2^2 + H_3^2$.*

Remark 3. The solutions of Euler equations are called **polhodes** in classical mechanics. The limit case with two or three equal principal moments of inertia can be treated similarly. Also, as pointed in [9], the above calculations hold for every left-invariant Hamiltonian $H = f(H_1, H_2, H_3)$ on $SO(3)$. In particular the SR-case is derived next as a limit of the Riemannian case.

3.2. **The sub-Riemannian case.** Setting $u_1 = \varepsilon v_1$, with $\varepsilon \to 0$, one gets a control system with two inputs only. Since $u_i = H_i/I_i$ this is equivalent to $I_1 \to +\infty$. Then

$$H_e = \frac{1}{2}\left(\frac{H_2^2}{I_2} + \frac{H_3^2}{I_3}\right)$$

with the corresponding Euler-Lagrange equation. This is the model for left-invariant SR-metrics on $SO(3)$ depending upon one parameter $k^2 = I_2/I_3$.

Once the Euler equation is integrated the next step is to parameterize the solution. It relies on the following general property [9].

Proposition 5. *For every left-invariant Hamiltonian $H_n = f(H_1, H_2, H_3)$ the full systems is integrable by quadratures using the first integrals: the Hamiltonian H and the Hamilonian lift $\xi(A_i R)$ of the* ***right****-invariant vector fields $A_i R$.*

Note that he additional first integrals are simply deduced from Noether theorem in Hamiltonian form.

3.3. **Explicit integration and Euler angles.** Euler angles are introduced on $SO(3)$ to complete the computations. They are denoted ϕ_1, ϕ_2, ϕ_3 and defined using the following convention:

$$R = (\exp \phi_1 A_3)(\exp A_2 \phi_2)(\exp \phi_3 A_3).$$

As it is shown in [9], the angles ϕ_2 and ϕ_3 can be founded using the relations

$$H_1 = -|H| \sin\phi_2 \cos\phi_3, \;\; H_2 = |H| \sin\phi_2 \sin\phi_3, \;\; H_3 = -|H| \cos\phi_2,$$

while ϕ_1 is computed by integrating the equation

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}t} = \frac{|H|\left(\sin\phi_3\frac{\partial H_n}{\partial H_2} - \cos\phi_3\frac{\partial H_n}{\partial H_1}\right)}{H_2\sin\phi_3 - H_1\cos\phi_3} = \frac{|H|\left(H_1\frac{\partial H_n}{\partial H_1} + H_2\frac{\partial H_n}{\partial H_2}\right)}{H_1^2 + H_2^2} \tag{4}$$

The following proposition is useful for the computations.

Proposition 6. *Using Euler angles the Hamiltonian* $H_e = \frac{1}{2}\left(\frac{H_1^2}{I_1} + \frac{H_2^2}{I_2} + \frac{H_3^2}{I_3}\right)$ *takes the form*

$$\begin{aligned} H_e =& \frac{1}{2I_1}\left(p_{\phi_2}\sin\phi_3 - \frac{\cos\phi_3}{\sin\phi_2}\left(p_{\phi_1} - p_{\phi_3}\cos\phi_2\right)\right)^2 \\ &+ \frac{1}{2I_2}\left(p_{\phi_2}\cos\phi_3 + \frac{\sin\phi_3}{\sin\phi_2}\left(p_{\phi_1} - p_{\phi_3}\cos\phi_2\right)\right)^2 + \frac{1}{2I_3}p_{\phi_3}^2. \end{aligned}$$

3.4. **The Serret-Andoyer variables and the associated metric.** Euler-Poinsot Hamiltonian can be computed in the symplectic Andoyer variables, see [7]. The moment of inertia are oriented according to $I_1 > I_2 > I_3$ and the Andoyer variables are denoted by (g, k, l, G, K, L) where G, K, L denote the canonical impulses associated to (g, k, l) . They are defined by

$$H_1 = \sqrt{G^2 - L^2}\sin l, \quad H_2 = \sqrt{G^2 - L^2}\cos l, \quad H_3 = L.$$

Hence $G = |H|$ and the Hamiltonian takes the form

$$H_e(g, k, l, G, K, L) = \frac{1}{2}\left(\frac{\sin^2 l}{I_1} + \frac{\cos^2 l}{I_2}\right)\left(G^2 - L^2\right) + \frac{L^2}{2I_3} \tag{5}$$

The complete relations between Andoyer variables and Euler angles are:

$$\begin{aligned} p_{\phi_1} &= K \\ p_{\phi_3} &= L \\ p_{\phi_2}\sin\phi_2 &= -\frac{1}{G}\sqrt{G^2 - L^2}\sqrt{G^2 - K^2}\sin g \\ \cos\phi_2 &= \frac{1}{G^2}\left(KL - \sqrt{G^2 - L^2}\sqrt{G^2 - K^2}\cos g\right) \\ \phi_1 &= k + \arctan\left(\frac{L - K\cos\phi_2}{p_{\phi_2}\sin\phi_2}\right) \\ \phi_2 &= l + \arctan\left(\frac{K - L\cos\phi_2}{p_{\phi_2}\sin\phi_2}\right) \end{aligned}$$

In the sequel we use the following notations for the Andoyer representation

$$H_e = \frac{1}{2}\left[\left(p_x^2 - p_y^2\right)\left(A\sin^2 y + B\cos^2 y\right) + Cp_y^2\right] \tag{6}$$

where

$$p_x = G = |H|, \quad y = l = \arctan\left(\frac{H_1}{H_2}\right), \quad z = 2\left(A\sin^2 y + B\cos^2 y\right),$$
$$A = I_1^{-1}, \quad B = I_2^{-1}, \quad C = I_3^{-1},$$

and the Hamiltonian takes the form

$$H_e(x, y, w, p_x, p_y, p_w) = \frac{1}{2}\left[\frac{z}{2}p_x^2 + \left(C - \frac{z}{2}\right)p_y^2\right],$$

where $A < B < C$, $z \in [2A, 2B]$, $C - \frac{z}{2} > 0$.

Thus H_e is associated to the so-called Riemannian **Serret-Andoyer metric**

$$g_a = \frac{2}{z}\mathrm{d}x^2 + \frac{2}{(2C-z)}\mathrm{d}y^2 \tag{7}$$

Remark 4. This metric is not related to the original metric on $SO(3)$ since the symplectic transformation mixed state variables and impulses. In particular, conjugate points of both metrics are not in correspondance. Nevertheless we have the following.

Lemma 3.1. *Denote by L the fiber $T^*_\phi SO(3)$ in Euler angles representation $\phi = (\phi_1, \phi_2, \phi_3)$, then L is a Lagrangian manifold in Serret-Andoyer variables.*

Proof. This is clear since Andoyer variables are symplectic coordinates. □

3.5. **The pendulum representation.** The dynamics of rigid body is well understood using Euler-Poinsot interpretation. The polhodes describes the evolution of the angular velocity in the moving frame and are contained in algebraic curves intersecting the ellipsoids $AH_1^2 + BH_2^2 + CH_3^2 = c_1$ with the spheres of constant angular momentum $H_1^2 + H_2^2 + H_3^2 = c_2$. They form periodic curves except pair of separatrices contained in the planes $H_3 = \pm\sqrt{\frac{B-A}{C-B}}H_1$. Besides those remarkable properties they can be parameterized using Jacobi or Weierstrass elliptic functions, see [11]. In particular the Euler equation can be used to define the $\mathrm{cn} - \mathrm{dn}$ Jacobi elliptic functions. To complete the integration one uses Euler-Poinsot interpretation. According to this representation the end-point of the angular velocity in a fixed frame describes a curve in a plane orthogonal to the angular momentum. Such a curve is called an **herpolhode** in mechanics. Its description by an angle and its parameterization using the elliptic integral of the third kind Π is given in [11]. Altogether this gives a pseudo-periodic motion described by two frequencies.

Serret-Andoyer representation leads to a different geometric interpretation that we describe next.

Using $H_e(x, y, w, p_x, p_y, p_w) = \frac{1}{2}\left[\frac{z}{2}p_x^2 + \left(C - \frac{z}{2}\right)p_y^2\right]$ one gets the equations

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= p_x\left(A\sin^2 y + B\cos^2 y\right), & \frac{\mathrm{d}p_x}{\mathrm{d}t} &= 0, \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= p_y\left(C - A\sin^2 y - B\cos^2 y\right), & \frac{\mathrm{d}p_y}{\mathrm{d}t} &= (B-A)\left(p_x^2 - p_y^2\right)\sin y\cos y,
\end{aligned} \tag{8}$$

completed by the trivial equations

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}p_w}{\mathrm{d}t} = 0.$$

The reduced system (8) describes the evolution of the extremals of the Andoyer metric. One has $\frac{\mathrm{d}y}{\mathrm{d}t} = p_y(C - \frac{z}{2})$ and the isoenergetic curves $H_e = h$ gives us

$$\frac{z}{2}p_x^2 + \frac{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}{(C - \frac{z}{2})} = 2h,$$

where $h > 0$ since $H_e > 0$. By homogeneity one can set $h = \frac{1}{2}$ which amounts to parameterize the extremals by arc-length. The Hamiltonian function is π-periodic with respect to the y-variable. It verifies the following relations:

$$H_e(y, p_y) = H_e(y, -p_y), \quad H_e(y, p_y) = H_e(-y, p_y).$$

The equilibrium points are $p_y = 0$, $y = \frac{k\pi}{2}$. In the neighborhood of the point $y = p_y = 0$ the eigenvalues of the linearized system are solutions of $\lambda^2 = (B-A)(C-B)p_x^2$ and they are real since $A < B < C$ and in the neighborhood of $y = \frac{\pi}{2}$, $p_y = 0$ they are purely imaginary. In consequence in order to parameterize all phase trajectories in the plane (y, p_y) it is sufficient to integrate with $y(0) = \frac{\pi}{2}$. To resume we have.

Proposition 7. *The Euler-Poinsot motion projects in the (y, p_y) plane into a pendulum motion which can be ineterpreted on the cylinder $y \in [0, \pi]$, with a stable equilibrium at $y = \frac{\pi}{2}$ and an an unstable at $y = 0$. They exist two types of periodic trajectories:* ***oscillating*** *trajectories homotopic to zero and* ***rotating*** *trajectories. The non periodic trajectories are separatrices joining 0 to π and represent separating polhodes. The trajectories are reflectionally symmetric with respect to the two axes: $y = 0$ and $p_y = 0$.*

Remark 5. This representation gives two types of trajectories while Euler-Poinsot representation gives only one type of peudo-periodic trajectories.

4. Main results: polar form of the metric and integration. Description of the conjugate locus.

4.1. Polar form of the metric.

The Serret-Andoyer metric is given by

$$g_a = \frac{2}{z}\mathrm{d}x^2 + \frac{2}{(2C - z)}\mathrm{d}y^2, \quad z = 2\left(A\sin^2 y + B\cos^2 y\right).$$

To get the polar from one integrates the equation

$$\frac{\mathrm{d}y}{\sqrt{C - \left(A\sin^2 y + B\cos^2 y\right)}} = \mathrm{d}\varphi$$

with the initial condition $y(0) = \frac{\pi}{2}$. One sets $X = \sin y$, $y \in [-\frac{\pi}{2}, +\frac{\pi}{2})$ and we have

$$\begin{aligned}\int_{\varphi(0)}^{\varphi} \mathrm{d}\varphi &= \int_1^X \frac{\mathrm{d}X}{\sqrt{(1-X^2)\left((C-B)+(B-A)X^2\right)}} \\ &= \int_1^X \frac{\mathrm{d}X}{\alpha\sqrt{(1-X^2)\left(k^2X^2+k'^2\right)}}\end{aligned}$$

where $\alpha = \sqrt{C-A}$, $k^2 = \frac{B-A}{C-A}$, $1 > k > 0$, $k'^2 = 1 - k^2$. Taking $\varphi(0) = 0$, one has

$$-\alpha\varphi = \int_X^1 \frac{\mathrm{d}X}{\sqrt{(1-X^2)\left(k'^2+k^2X^2\right)}} = \mathrm{cn}^{-1}(X, k).$$

Hence

$$X = \sin y = \mathrm{cn}(-\alpha\varphi, k) = \mathrm{cn}(\alpha\varphi, k). \tag{9}$$

This yields

$$z = 2\left(A\sin^2 y + B\cos^2 y\right) = 2\left(A\,\mathrm{cn}^2(\alpha\varphi, k) + B\,\mathrm{sn}^2(\alpha\varphi, k)\right)$$

and

$$m(\varphi) = \frac{2}{z} = \frac{1}{A\,\mathrm{cn}^2(\alpha\varphi, k) + B\,\mathrm{sn}^2(\alpha\varphi, k)}$$

Hence we get the following proposition using the notation $x = \theta$.

Proposition 8. *The Serret-Andoyer metric g_a takes the polar normal form* $\mathrm{d}\varphi^2 + m(\varphi)\mathrm{d}\theta^2$ *where* $m(\varphi) = [A\,\mathrm{cn}^2(\alpha\varphi, k) + B\,\mathrm{sn}^2(\alpha\varphi, k)]^{-1} \in [I_2, I_1]$, $k^2 = \frac{B-A}{C-A}$, $\alpha = \sqrt{C-A}$. *It is reflectionally symmetric with respect to the equator* ($m(-\varphi) = m(\varphi)$).

4.2. **Geometric analysis.** The Hamiltonian associated to the metric is $H = \frac{1}{2}(p_\varphi^2 + \frac{p_\theta^2}{m(\varphi)})$ and parameterizing using $H = \frac{1}{2}$ one gets the mechanical system $(\frac{\mathrm{d}\varphi}{\mathrm{d}t})^2 + V(\varphi, p_\theta) = 1$ where the potential is given by

$$V(\varphi, p_\theta) = p_\theta^2 \left[A\,\mathrm{cn}^2(\alpha\varphi, k) + B\,\mathrm{sn}^2(\alpha\varphi, k)\right]$$

Lemma 4.1. *The potential is symmetric with respect to the equator* $\varphi = 0$*:* $V(\varphi) = V(-\varphi)$ *and is periodic of period* $\frac{2K(k)}{\alpha}$. *It has a minimum at* $\varphi = 0$ *given by* $p_\theta^2 A$ *and a maximum at* $\varphi = \frac{K}{\alpha}$ *given by* $p_\theta^2 B$ *and is monotonic on* $[0, \frac{K}{\alpha}]$.

To compute the extremals starting from the equator $\varphi = 0$ we proceed as follows. One can restrict to $p_\theta \geq 0$ and one must have $p_\theta \in [0, \sqrt{I_1}]$. We have two types of solutions associated to the pendulum representation :

- Rotating trajectories: for physical solution one has $p_\theta \in (\sqrt{I_3}, \sqrt{I_2})$ but to get a complete metric they are extended to $p_\theta \in (0, \sqrt{I_2})$.
- Oscillating trajectories: $p_\theta \in (\sqrt{I_2}, \sqrt{I_1})$.

Except the separatrices all the trajectories are periodic and due to the symmetry with respect to the equator $\varphi = 0$ it is sufficient to parameterize the following:

- For oscillating trajectories denote φ_+ the intersection of the potential with the level set 1. We parameterize the branch between $[0, \varphi_+]$.
- For rotating trajectories, one may assume $\varphi \in [0, \frac{K}{\alpha}]$.

4.3. **Parameterization.** One set $Y = \mathrm{sn}^2(\alpha\varphi, k)$ and for the increasing branch we integrate

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \sqrt{1 - p_\theta^2\,[A + (B-A)Y]}$$

Moreover

$$\mathrm{d}Y = 2\alpha\,\mathrm{sn}\,\mathrm{cn}\,\mathrm{dn}\,\mathrm{d}\varphi.$$

Using $\alpha = \sqrt{C-A}$, $k^2 = \frac{B-A}{C-A}$, $\mathrm{dn}^2 = 1 - k^2\,\mathrm{sn}^2$, one gets:

$$\mathrm{d}t = \frac{\mathrm{d}y}{2(B-A)p_\theta\sqrt{Y\,(1-Y)\left(\frac{1}{k^2} - Y\right)(Y_+ - Y)}} \tag{10}$$

where Y_+ denotes the root of $0 = 1 - p_\theta^2(A + (B-A)Y)$.

The two cases are distinguished by the position of this root. In the oscillating case where $p_\theta \in (\sqrt{I_2}, \sqrt{I_1})$ one has $Y_+ = \mathrm{sn}^2(\alpha\varphi_+, k)$ and for $p_\theta \in (0, \sqrt{I_2})$, $Y_+ > 1$. In the limit case $p_\theta^2 = \frac{1}{B}$, $Y_+ = 1$. Also observe that when $p_\theta^2 = \frac{1}{C} = I_3$, this root is given by $Y_+ = \frac{C-A}{B-A} = \frac{1}{k^2}$. Hence we recover the different physical cases in Euler-Poinsot easily.

To integrate, we use a specific homographic integration because the roots are explicit. Besides the geometric interest it is related to the uniformization of the computations of the conjugate locus in relation with [5]. We proceed as follows.

4.3.1. *Parameterization of oscillating trajectories.* We have for $p_\theta \in (\sqrt{I_2}, \sqrt{I_1})$ four distinct roots for the polynomial at the right-hand member of (10): $0 < Y_+ < 1 < \frac{1}{k^2}$ and for the non fixed roots we use the notation $Y_1 = Y_+$ and $Y_2 = \frac{1}{k^2}$ and the motion can be restricted to the interval $[0, Y_1]$. The equation (10) takes the form

$$\left(\frac{\mathrm{d}Y}{\mathrm{d}t}\right)^2 = -4p_\theta^2(B-A)^2 Y(Y-Y_1)(Y-1)(Y-Y_2) \tag{11}$$

and we set

$$\eta^2 = \frac{Y(1-Y_1)}{Y_1(1-Y)}. \tag{12}$$

and η is monotone increasing from 0 to 1. We have

$$Y = \frac{\eta^2 Y_1}{\eta^2 Y_1 + (1-Y_1)}, \quad \mathrm{d}Y = \frac{2\eta Y_1(1-Y_1)\mathrm{d}\eta}{[\eta^2 Y_1 + (1-Y_1)]^2} \tag{13}$$

and after simplication the equation (11) takes the normal form

$$\left(\frac{\mathrm{d}\eta}{\mathrm{d}t}\right)^2 = p_\theta^2(B-A)^2(Y_2-Y_1)(1-\eta^2)(m'^2+m^2\eta^2) \tag{14}$$

with

$$m^2 = \frac{Y_1(Y_2-1)}{Y_2-Y_1}, \quad m'^2 = 1-m^2.$$

Denoting

$$M^2 = p_\theta^2(B-A)^2(Y_2-Y_1) \tag{15}$$

one gets the parameterization

$$\eta(t) = \mathrm{cn}(Mt+\psi_0, m).$$

Therefore we have

$$Y = \frac{Y_1 \,\mathrm{cn}^2(Mt+\psi_0,m)}{[Y_1\,\mathrm{cn}^2(Mt+\psi_0,m)+(1-Y_1)]} = \frac{Y_1\,\mathrm{cn}^2(Mt+\psi_0,m)}{1-Y_1\,\mathrm{sn}^2(Mt+\psi_0,m)}. \tag{16}$$

To integrate the θ-variable one uses the following relations [11], p. 68.

$$\int_0^u \frac{\alpha+\beta\,\mathrm{sn}^2 u}{\lambda+\mu\,\mathrm{sn}^2 u}\mathrm{d}u = \frac{\alpha}{\lambda}u + \frac{1}{\lambda^2}(\beta\lambda-\alpha\mu)\int \frac{\mathrm{sn}^2 u}{1+\nu\,\mathrm{sn}^2 u}\mathrm{d}u \tag{17}$$

where $\nu = \frac{\mu}{\lambda}$. To compute the integral in the right-hand member, one introduces the complex parameter a defined by

$$\mathrm{sn}^2 a = -\frac{\nu}{m^2} \tag{18}$$

and the elliptic integral of the third kind

$$\Pi(u,a,m) = \int_0^u \frac{m^2\,\mathrm{sn}\,a\,\mathrm{cn}\,a\,\mathrm{dn}\,a\,\mathrm{sn}^2 v}{1-m^2\,\mathrm{sn}^2 a\,\mathrm{sn}^2 v}\mathrm{d}v \tag{19}$$

Therefore

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = p_\theta[A+(B-A)Y] = p_\theta\left[A + \frac{(B-A)Y_1\,\mathrm{cn}^2(Mt+\psi_0,m)}{(1-Y_1\,\mathrm{sn}^2(Mt+\psi_0,m))}\right].$$

One sets

$$\alpha = 1, \quad \beta = -1, \quad \lambda = \frac{1}{Y_1}, \quad \mu = -1$$

and with $\theta(0) = 0$, we obtain

$$\theta(t) = p_\theta \left[At + (B - A) \left[\frac{\alpha}{\lambda} t + \frac{1}{\lambda^2}(\beta\lambda - \alpha\mu) I \right] \right] \tag{20}$$

where

$$I = \int_0^t \frac{\operatorname{sn}^2 u}{1 + \nu \operatorname{sn}^2 u} \mathrm{d}t = \frac{1}{M} \int_{\psi_0}^u \frac{\operatorname{sn}^2 u}{1 + \nu \operatorname{sn}^2 u} \mathrm{d}u, \quad u = Mt + \psi_0$$

which can be represented with (19)

$$\int_{\psi_0}^u \frac{\operatorname{sn}^2 u}{1 + \nu \operatorname{sn}^2 u} \mathrm{d}u = \frac{\Pi(u, a, m)}{m^2 \operatorname{sn} a \operatorname{cn} a \operatorname{dn} a}.$$

The parameter a is given by

$$\operatorname{sn}^2 a = \frac{Y_1}{m^2} = \frac{Y_2 - Y_1}{Y_2 - 1} > 1.$$

This gives the complete parameterization of the Euler-Poinsot rigid body motion using the Serret-Andoyer variables since the remaining components are defined by $p_\varphi = \frac{\mathrm{d}\varphi}{\mathrm{d}t}$ and the additional first integrals.

4.3.2. *Parameterization of rotating trajectories.* In this case $Y_+ > 1$ and we use the transformation

$$\eta^2 = \frac{Y(Y_1 - 1)}{Y_1 - Y}$$

and for $Y = 0$, $\eta = 0$ and for $Y = 1$, $\eta = 1$. We omit the details of the computations. We have two cases $p_\theta \in (\sqrt{I_3}, \sqrt{I_2})$ and $p_\theta \in (0, \sqrt{I_3})$, the case $p_\theta = \sqrt{I_3}$ corresponding to the collision of Y_+ with $\frac{1}{k^2}$. Geometrically, it has no effect since we consider onty the part of the trajectory between 0 and 1.

4.4. **Construction of the conjugate locus.** We shall use the framework of Section 2. One needs some preliminary computations.

Lemma 4.2. *The Gauss curvature of the Serret-Andoyer metric is given by*

$$G = \frac{(A + B + C)(z - z_-)(z - z_+)}{z^2}$$

with $z_\pm = \frac{2(AB+AC+BC \pm \sqrt{(AB+AC+BC)^2 - 3ABC(A+B+C)})}{A+B+C}$. *It has a positive maximum* $\frac{(B-A)(C-A)}{A}$ *at the equator.*

Proposition 9. *For oscillating trajectories where* $I_2 < p_\theta^2 < I_1$, *the first return mapping of the Serret-Andoyer metric is a strictly convex monotone decreasing function.*

We immediatly deduce the conjugate locus at the equator. For rotating trajectories there is no conjugate points. For oscillating trajectory and p_θ positive the conjugate locus is a curve formed by two symmetric branches with respect to the equator and the branch in $\varphi \leq 0$ is starting from the horizontal cusp point at the equator when $p_\theta^2 \to I_1$ to form when $p_\theta^2 \to I_2$ a branch asymptotic to the parallel direction $\varphi = -\frac{K(k)}{\alpha}$.

Remark 6. This can be compared with the oblate ellipsoid of revolution. The astroidal caustic ending at $\theta = \pi$ when $p_\theta \to 0$ is replaced by an asymptotic branch when $p_\theta^2 \to I_2$ due to the separatrix in the pendulum.

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