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RESEARCH ARTICLE

Differential continuation for regular optimal control problems

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Regular control problems in the sense of the Legendre condition are defined, and second order necessary and sufficient optimality conditions in this class are reviewed. Adding a scalar homotopy parameter, differential pathfollowing is introduced. The previous sufficient conditions ensure the definiteness and regularity of the path. The role of AD for the implementation of this approach is discussed, and two examples excerpted from quantum and space mechanics are detailed.

Keywords: optimal control; second order conditions; conjugate points; differential homotopy; automatic differentiation

AMS Subject Classification: 49K15; 49-04

Introduction

In [17], automatic differentiation allowed to assemble the two-point boundary value problem embodying the first order necessary optimality condition of an optimal control problem. ADIFOR [6] was used, and sensitivities were computed using AD again, in connection with primitive second order optimality conditions. This approach was given a full treatment with the development of the `cotcot` code [9]. Given the data defining the control problem, a set of *Matlab* routines was generated through AD so as to solve the problem by single shooting and to systematically check conjugate point second order conditions [10]. The present paper aims at presenting the next step, that is the addition of a layer automating differential pathfollowing for optimal control problems depending on one scalar homotopic parameter: The differential equation that embodies the continuation procedure is automatically generated and numerical integration performed. The resulting code [15] relies crucially on AD to compile the definition of the parametric control problem into a collection of *Matlab* MEX-files. In particular, the core of the method is to replace shooting by the mere integration of an AD-defined ODE. Differential pathfollowing replaces Newton solving—except for the computation of the starting point where shooting is still needed—, and provides numerical checks all along the path of optimal control second order sufficient conditions.

The paper is organized in five sections. The first one settles the mathematical framework for second order necessary and sufficient conditions. The infinite-dimensional setting together with the dynamical features typical of optimal control of ODEs have some

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peculiarities, and we give a more or less self-contained overview of the necessary part along the lines of [1]. Section two introduces differential homotopy for the regular class of optimal control problems just defined. The role of AD and other algorithmic issues are reviewed in section three. Finally, two examples coming from ongoing research projects are treated. One in dissipative quantum control, the other in space mechanics. In the first case, homotopy explores the change of solutions in a range of physical parameters, while in the latter pathfollowing is used to regularize and simplify the L^1-minimization problem considered.

1. Second order conditions for regular optimal control problems

Let us consider the following optimal control problem: Minimize the integral or *Lagrange* cost functional

$$\int_0^{t_f} f^0(x(t), u(t))\,\mathrm{d}t \to \min$$

subject to differential constraints

$$\dot{x}(t) = f(x(t), u(t)), \quad u(t) \in U, \quad t \in [0, t_f],$$

and boundary conditions

$$x(0) = x_0, \quad x(t_f) = x_f.$$

The state, x, lives on an n-dimensional manifold[1] X so, in local coordinates, $x(t) \in \mathbf{R}^n$. The control, u, belongs to an m-dimensional manifold U. Being without boundary, U has charts with open domains in $\mathbf{R}^m$ and $u(t) \in \mathbf{R}^m$, locally. The latter is a strong restriction as typical control problems involve control constraints, that is manifolds U with boundary.[2] Such an example is addressed in §For the sake of simplicity, the positive final time t_f is assumed to be fixed. The data (f^0, f) is smooth[3] in (x, u). We seek a measurable essentially bounded control so, in coordinates, $u \in \mathrm{L}^\infty_m([0, t_f]) = \mathrm{L}^\infty([0, t_f], \mathbf{R}^m)$. In order to assess first and second order necessary optimality conditions, let $\overline{u}$ be an optimal control, and let $\overline{x}$ be the associated Lipschitz optimal trajectory defined on the whole interval $[0, t_f]$. We define the *augmented system*, which amounts to incorporating the cost into the state. Set $\hat{x} = (x^0, x) \in \mathbf{R}^{n+1}$. Almost for all t,

$$\dot{\hat{x}}(t) = \hat{f}(\hat{x}(t), u(t)) \quad \text{with} \quad \hat{f}(\hat{x}, u) = (f^0, f)(x, u).$$

Define $\hat{x}(0) = (0, x_0)$, so the cost is retrieved as one of the components of the augmented state. From now on, we drop the $\hat{}$ on variables to simplify notations, keeping in mind that in our context $x(t) \in \mathbf{R}^n$ stands for $(x^0(t), x(t)) \in \mathbf{R}^{n+1}$, and f for (f^0, f). What follows is a very partial account of [1, chap. 20-21], without the differential-geometric machinery. Among many others, see also [11, 25, 28] and references therein.

For any $s \in [0, t_f]$, the *endpoint mapping*

$$F_s : u \mapsto x(s, u) \in \mathbf{R}^n$$

[1] Unless otherwise specified manifolds are supposed to be $\mathscr{C}^\infty$-smooth.
[2] This does not exclude applications with bounded controls, though, as U may be a compact submanifold. See also [17] ($U = \mathbf{S}^1$) and [16] ($U = \mathbf{S}^2$).
[3] That is $\mathscr{C}^\infty$-smooth.

that maps a control to the solution of

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0,$$

is well defined and smooth in a neighbourhood in $\mathrm{L}^\infty_m([0, s])$ of $\overline{u}$ by virtue of the implicit function theorem, and

$$F'_s(\overline{u})\,\delta u = \delta x(s), \quad \delta u \in \mathrm{L}^\infty_m([0, s]),$$

where δx is the solution of

$$\delta\dot{x}(t) = \partial_x f(\overline{x}(t), \overline{u}(t))\,\delta x(t) + \partial_u f(\overline{x}(t), \overline{u}(t))\,\delta u(t), \quad \delta x(0) = 0. \tag{1}$$

The first order necessary condition is obtained asserting that F_{t_f} cannot be locally open[1] at $\overline{u}$. We recall the following standard (nonlinear) open mapping result.

THEOREM 1.1 *Let $F : E \to \mathbf{R}^n$ be a smooth[2] mapping on the Banach space E. Let u be a non-critical point ($F'(u)$ is onto), then F is locally open at u.*

As result, $F'_{t_f}(\overline{u})$ cannot be onto, and there is a nonzero covector $\lambda \in (\mathbf{R}^n)^*$ such that $\overline{\lambda}F'_{t_f}(\overline{u}) = 0$. This is Lagrange rule which gives Pontryagin maximum principle in weak form. Let $\overline{p}$ be the Lipschitz solution (valued in $(\mathbf{R}^n)^*$) of the *adjoint equation*

$$\dot{p}(t) = -p(t)\,\partial_x f(\overline{x}(t), \overline{u}(t)), \quad p(t_f) = \overline{\lambda}.$$

The triple $(\overline{x}, \overline{u}, \overline{p})$ is called an *extremal*. For any $\delta u \in \mathrm{L}^\infty_m([0, t_f])$,

$$0 = \overline{\lambda}F'_{t_f}(\overline{u})\,\delta u = \overline{p}(t_f)\,\delta x(t_f).$$

Integrating by parts,

$$\int_0^{t_f} \partial_u H(\overline{x}(t), \overline{u}(t), \overline{p}(t))\,\delta u(t)\,\mathrm{d}t = 0$$

where $H(x, u, p) = pf(x, u)$ is the Hamiltonian function. Accordingly, $\partial_u H(\overline{x}(t), \overline{u}(t), \overline{p}(t)) = 0$ almost everywhere on $[0, t_f]$. Further optimality conditions are obtained using the following second order open mapping theorem.

THEOREM 1.2 *Let $F : E \to \mathbf{R}^n$ be a smooth mapping on the Banach space E. Let u be a corank one critical point of F (*$\operatorname{codim} \operatorname{Im} F'(u) = 1$*), and let $0 \neq \lambda \in (\mathbf{R}^n)^*$ belong to $[\operatorname{Im} F'(u)]^\perp$. If $\lambda F''(u)$ is sign-indefinite on* $\operatorname{Ker} F'(u)$*, then F is locally open at u.*

The bilinear form $\lambda F''(u) \in \mathscr{L}(E, \mathscr{L}(E, \mathbf{R})) \simeq \mathscr{L}_2(E, E; \mathbf{R})$ is called the *intrinsic second order derivative* of F and is defined up to a scalar in the corank one case. In our situation, $\overline{\lambda}F''_{t_f}(\overline{u})$ has to be sign-semidefinite on $\operatorname{Ker} F'_{t_f}(\overline{u})$ for local optimality to hold. Given $s \in [0, t_f]$, define the symmetric (Schwarz) bilinear form $B_s = \overline{\lambda}F''(\overline{u})$. By the implicit function theorem, for δu and δv in $\mathrm{L}^\infty_m([0, s])]$,

$$F''_s(\overline{u})\,(\delta u, \delta v) = \delta_2 x(s)$$

[1] A mapping between topological spaces is locally open at a point if it sends neighbourhood of the point onto neighbourhoods of its image.
[2] That is $\mathscr{C}^\infty$-smooth.

with

$$\delta_2\dot{x}(t) = \partial_x f[t]\,\delta_2 x(t) + \partial^2_{xx} f[t]\,(\delta x(t), \delta y(t)) + \partial^2_{xu} f[t]\,(\delta x(t), \delta v(t)) \\ + \partial^2_{ux} f[t]\,(\delta u(t), \delta y(t)) + \partial^2_{uu} f[t]\,(\delta u(t), \delta v(t)), \quad \delta_2 x(0) = 0,$$

and δx (*resp.* δy) associated with δu (*resp.* δv) according to (1). The argument $[t]$ in partial derivatives of f stands for $(\overline{x}(t), \overline{u}(t))$ for notational convenience. Integrating by parts as before,

$$B_s\,(\delta u, \delta v) = \int_0^s [\,\partial^2_{xx} H[t]\,(\delta x(t), \delta y(t)) + \partial^2_{xu} H[t]\,(\delta x(t), \delta v(t)) \\ + \partial^2_{ux} H[t]\,(\delta u(t), \delta y(t)) + \partial^2_{uu} H[t]\,(\delta u(t), \delta v(t))\,]\,\mathrm{d}t$$

the argument $[t]$ standing now for $(\overline{x}(t), \overline{u}(t), \overline{p}(t))$. Since $\overline{u}$ is essentially bounded, B_s remains continuous on $\mathrm{L}^\infty_m([0, s])$ endowed with the weaker L^2-norm,[1] and can be continuously extended to $\mathrm{L}^2_m([0, s])$ by density. We set

$$K_s = \overline{\operatorname{Ker} F'_s(\overline{u})}^{\|.\|_2},$$

and remark that the sign-definiteness of B_s on $\operatorname{Ker} F'_s(\overline{u})$ is equivalent to the sign-semidefiniteness on K_s of the extension of B_s to L^2_m. Using the embedding of the original space into a Hilbert one to check sign-definiteness is referred to as *two-norm discrepancy* [25]. On this Hilbert space, $B_s \in \mathscr{L}(\mathrm{L}^2_m, \mathscr{L}(\mathrm{L}^2_m, \mathbf{R})) \simeq \mathscr{L}(\mathrm{L}^2_m)$ is identified with a self-adjoint operator. Similarly, the restriction $B_{s|K_s}$ of the bilinear form to K_s is identified with $j^*_{K_s} B_s j_{K_s} \in \mathscr{L}(K_s)$ where j_{K_s} is the canonical injection $K_s \hookrightarrow \mathrm{L}^2_m([0, s])$ (and $j^*_{K_s}$ its adjoint). The following assumption, known as the *(strong) Legendre condition*, will ensure the negative definiteness of B_s on all $\mathrm{L}^2_m([0, s])$ for small enough s:

$$\nabla^2_{uu} H(\overline{x}(t), \overline{u}(t), \overline{p}(t)) \leq -\alpha I_m, \quad t \in [0, t_f],$$

for some $\alpha > 0$. Extremals verifying the Legendre condition are called *regular*.

Proposition 1.3 *Under the Legendre condition, B_s is identified with the Fredholm operator $(-I + R_s)$ on $\mathrm{L}^2_m([0, s])$, R_s a compact operator, and is negative definite for small s.*

Proof. The Legendre condition implies that

$$-\int_0^s \partial^2_{uu} H[t]\,(\delta u(t), \delta v(t))\,\mathrm{d}t$$

is an equivalent scalar product on $\mathrm{L}^2_m([0, s])$. In this identification, $B_s = -I + R_s$ where R_s is associated with the bilinear part

$$\int_0^s [\,\partial^2_{xx} H[t]\,(\delta x(t), \delta y(t)) + \partial^2_{xu} H[t]\,(\delta x(t), \delta v(t)) + \partial^2_{ux} H[t]\,(\delta u(t), \delta y(t))\,]\,\mathrm{d}t. \qquad (2)$$

Now, $\delta x = T\,\delta u$ where $T = j \circ T_0$ is the composition of the canonical injection $j : \mathrm{H}^1_m([0, s]) \hookrightarrow \mathrm{L}^2_m([0, s])$ which is a compact operator [14], and of $T_0 \in \mathscr{L}(\mathrm{L}^2_m, \mathrm{H}^1_m)$ which maps δu to δx. Then T is compact on L^2_m, and so is R_s as there is at least one left or right

[1] The application remains continuous though the topology is weakened because $\mathrm{L}^2([0, s])$ is a topological module over $\mathrm{L}^\infty([0, s])$.

composition with T in each term of (2). Moreover, T can be represented as the integral Hilbert-Schmidt operator

$$(T\,\delta u)(t) = \int_0^t \Phi(t-\sigma)\,\partial_u f[\sigma]\,\delta u(\sigma)\,\mathrm{d}\sigma$$

where Φ is the fundamental solution of $\delta\dot{x}(t) = \partial_x f[t]\,\delta x(t)$. Then, denoting $|.|$ the equivalent norm on L^2_m, one has the estimate

$$B_s\,(\delta u, \delta u) = -|\delta u|^2 + O(s)|\,\delta u|^2, \quad s \to 0,$$

and the negative definiteness for small s follows. ■

Clearly, if $s \leq t$, $B_{t|K_t} < 0$ implies $B_{s|K_s} < 0$, hence the following definition. The *first conjugate time* along the extremal, t_{1c}, is

$$t_{1c} = \sup\{s > 0 \mid B_{s|K_s} < 0\}.$$

Proposition 1.4 *(see [1]) Let $t_{1c} \in (0, t_f]$ for a regular extremal. Then $B_{t_{1c}|K_{t_{1c}}} \in \mathscr{L}(K_{t_{1c}})$ has a nontrivial kernel.*

Conjugate times are more generally defined as times t_c such that $B_{t_c|K_{t_c}}$ is degenerate. Points $\overline{x}(t_c)$ are then referred to as conjugate points. Under the Legendre condition, the equation $\partial_u H(x,u,p) = 0$ can be solved in the neighbourhood of the extremal, yielding the smooth implicit function $u(x,p)$. Plugging $u(x,p)$ into H defines the *true Hamiltonian*, $H(x,p) = H(x,u(x,p),p)$ (still denoted H, the number of arguments avoiding any ambiguity). Remarkably,

$$\begin{aligned}
\dot{\overline{x}}(t) &= \partial_p H(\overline{x}(t),\overline{u}(t),\overline{p}(t)) = \partial_p H(\overline{x}(t),\overline{p}(t)),\\
\dot{\overline{p}}(t) &= -\partial_x H(\overline{x}(t),\overline{u}(t),\overline{p}(t)) = -\partial_x H(\overline{x}(t),\overline{p}(t)),
\end{aligned}$$

which we write more compactly $\dot{\overline{z}}(t) = \overrightarrow{H}(\overline{z}(t))$, with $z = (x,p)$ and $\overrightarrow{H} = (\nabla_p H, -\nabla_x H)$ the symplectic gradient. A solution of the equation linearized along $\overline{z}$, the *Jacobi* equation,

$$\delta\dot{z}(t) = \overrightarrow{H}'(\overline{z}(t))\,\delta z(t),$$

is called a *Jacobi field*. Such a field $\delta z = (\delta x, \delta p)$ is said to be vertical at time t whenever $\delta x(t) = 0$. These notions allow to characterize and compute conjugate times.

Proposition 1.5 *An instant $s \in (0, t_f]$ along a regular extremal is a conjugate time if and only if there exists a Jacobi field $\delta z = (\delta x, \delta p)$ vertical at 0 and s such that $\delta x \not\equiv 0$ on $[0, t_c]$.*

Proof. Let $\delta u \in \mathrm{Ker}\; B_{s|K_s}$, $\delta u \neq 0$. Then $B_s\,\delta u$ must belong to $K_s^\perp$. But K_s which is the closure in L^2_m of $\mathrm{Ker}\, F_s'(\overline{u})$ is also equal to $\mathrm{Ker}\, G_s$, $G_s \in \mathscr{L}(\mathrm{L}^2_m, \mathbf{R}^n)$ being the extension of $F_s'(\overline{u})$ to L^2_m (such an extension exists and remains continuous because $\overline{u}$ is bounded). Indeed,

Lemma 1.6 *Let F be an everywhere dense subspace of a normed space, E, and let G be finite-dimensional. Let $u \in \mathscr{L}(F,G)$, and let $v \in \mathscr{L}(E,G)$ be its continuous linear extension. Then* $\mathrm{Ker}\, v = \overline{\mathrm{Ker}\, u}$.

Proof (of lemma). One has $F = \mathrm{Ker}\, u \oplus H$ where $H \simeq \mathrm{Im}\, u$ and the direct sum is algebraic. As G is finite-dimensional, the image of u is closed so, by definition of v, $\mathrm{Im}\, u = \mathrm{Im}\, v$.

Hence, $E = \operatorname{Ker} v \oplus H$ algebraically. Both $\operatorname{Ker} v$ and H are closed, the latter being finite-dimensional, so the direct sum is also topological in E. Readily, $\operatorname{Ker} u \subset \operatorname{Ker} v$, so $\operatorname{Ker} u \times H \subset \operatorname{Ker} v \times H$. Since $E = \overline{F}$, $\operatorname{Ker} v \times H = \overline{\operatorname{Ker} u \times H} = \overline{\operatorname{Ker} u} \times \overline{H} = \overline{\operatorname{Ker} u} \times H$, whence the conclusion. ■

Thus, $K_s^\perp = (\operatorname{Ker} G_s)^\perp = \operatorname{Im} G_s^*$, and there exists $\xi \in \mathbf{R}^n$ such that, for any δv in $\mathrm{L}^2_m([0, s])$,

$$B_s\,(\delta u, \delta v) = (G_s^*\xi | \delta v)_{\mathrm{L}^2_m} = (\xi | G_s\,\delta v)_{\mathbf{R}^n} = \delta p(s)\,\delta y(s)$$

with δp the covector valued solution of

$$\delta\dot p(t) = -\delta p(t)\,\partial_x f[t] - \overline{p}(t)\,\partial^2_{xx} f[t]\,\delta x(t) - \overline{p}(t)\,\partial^2_{ux} f[t]\,\delta u(t), \quad \delta p_i(s) = -\xi_i, \quad i = 1, n,$$

and δx (*resp.* δy) associated to δu (*resp.* δv) according to (1). Integrating by parts and compensating terms from both sides, one gets

$$\int_0^s [\,\partial^2_{uu} H[t]\,(\delta u(t), \delta v(t)) + \partial^2_{xu} H[t]\,(\delta x(t), \delta v(t)) + \partial^2_{pu} H[t]\,(\delta p(t), \delta v(t))\,]\,\mathrm{d}t = 0$$

for arbitrary δv so that on $[0, s]$

$$\delta u(t) = -(\nabla^2_{uu} H[t])^{-1}(\nabla^2_{xu} H[t]\,\delta x(t) + \nabla^2_{pu} H[t]\,\delta p^T(t)).$$

Plugging this expression of δu into the differential equations of δx and δp, one verifies that $\delta z = (\delta x, \delta p)$ is a Jacobi fied. As δu belongs to $K_s = \operatorname{Ker} G_s$, $\delta x(s) = G_s\,\delta u = 0$ and the field is vertical at 0 and s. Besides, if $\delta x \equiv 0$,

$$0 = B_s\,(\delta u, \delta u) = \int_0^s \partial^2_{uu} H[t]\,(\delta u(t), \delta u(t))\,\mathrm{d}t \leq -\alpha \|\delta u\|^2_{\mathrm{L}^2_m}$$

by Legendre condition, so δu would be zero, which is not. Conversely, let $\delta z = (\delta x, \delta p)$ be a Jacobi field vertical at 0 and s, $\delta x \not\equiv 0$. Setting

$$\delta u(t) = -(\nabla^2_{uu} H[t])^{-1}(\nabla^2_{xu} H[t]\,\delta x(t) + \nabla^2_{pu} H[t]\,\delta p^T(t)),$$

one has

$$\begin{aligned}\delta\dot x &= \partial_x(\nabla_p H)(\overline{x}(t), \overline{p}(t))\,\delta x(t) + \partial_p(\nabla_p H)(\overline{x}(t), \overline{p}(t))\,\delta p(t)\\ &= \partial_x f[t]\,\delta x(t) + \partial_u f[t]\,\delta u(t), \quad \delta x(0) = 0.\end{aligned} \tag{3}$$

In particular, $\delta u \in K_s$ as $\delta x(s) = 0$. For $\delta v \in K_s$, using the definition of δu,

$$B_s\,(\delta u, \delta v) = \int_0^s [\,\partial^2_{xx} H[t]\,(\delta x(t), \delta v(t)) + \partial^2_{ux} H[t]\,(\delta u(t), \delta y(t)) - \partial^2_{pu} H[t]\,(\delta p(t), \delta v(t))\,]\,\mathrm{d}t.$$

As

$$\begin{aligned}\delta\dot p &= \partial_x(-\nabla_x H)(\overline{x}(t), \overline{p}(t))\,\delta x(t) + \partial_p(-\nabla_x H)(\overline{x}(t), \overline{p}(t))\,\delta p(t)\\ &= -\delta p(t)\,\partial_x f[t] - \partial^2_{xx} f[t]\,\delta x(t) - \partial^2_{uu} f[t]\,\delta u(t),\end{aligned}$$

an integration by parts give

$$B_s(\delta u, \delta v) = -\delta p(t)\,\delta y(t)]_0^s = 0$$

since $\delta v \in K_s$ (so $\delta y(0) = \delta y(s) = 0$). Therefore, δu belongs to Ker $B_{s|K_s}$. As $\delta u = 0$ would imply $\delta x \equiv 0$ by virtue of (3), we conclude that $B_{s|K_s}$ has indeed a non-trivial kernel. ■

We finally formulate the second order necessary condition for regular extremals in the corank one analytic case.[1]

THEOREM 1.7 *[1, 28] For a regular extremal associated with an analytic corank one optimal control, the absence of conjugate times on* $(0, t_f)$ *is necessary for* L^∞*-local optimality.*

Proof. Suppose there exists a conjugate time $t_c \in (0, t_f)$. By definition, the kernel of $B_{t_c|K_{t_c}}$ is not trivial so one can find $0 \neq \delta u \in K_{t_c}$ such that $B_{t_c}(\delta u, \delta u) = 0$. Let us define $\widetilde{\delta u} \in K_{t_f}$ as the extension of δu by 0 on $[t_c, t_f]$, and check that it cannot belong to Ker $B_{t_f|K_{t_f}}$. If this were the case, one could construct as in the proof of Proposition 1.5 a Jacobi field $\widetilde{\delta z} = (\widetilde{\delta x}, \widetilde{\delta p})$ vertical at 0 and t_f, such that $\widetilde{\delta x} \not\equiv 0$ ($\delta u \neq 0$, so $\widetilde{\delta u} \neq 0$). But as

$$\dot{\widetilde{\delta x}}(t) = \partial_x f[t]\,\widetilde{\delta x}(t) + \partial_u f[t]\,\widetilde{\delta u}(t), \quad \widetilde{\delta x}(t_f) = 0,$$

$\widetilde{\delta x}$ would identically vanish on $[t_c, t_f]$ (nonempty since $t_c < t_f$). Under the assumption that $\overline{u}$ is analytic, the Jacobi equation has analytic data and so analytic solutions. In particular, $\widetilde{\delta x}$ would identically vanish over $[0, t_f]$, whence the contradiction. As $\widetilde{\delta u} \notin$ Ker $B_{t_f|K_{t_f}}$, one can find $\widetilde{\delta v} \in K_{t_f}$ such that $B_{t_f}(\widetilde{\delta u}, \widetilde{\delta v}) \neq 0$. Clearly, $B_{t_f}(\widetilde{\delta u}, \widetilde{\delta u}) = B_{t_c}(\delta u, \delta u) = 0$, so B_{t_f} takes opposite signs on the subspace generated by $\widetilde{\delta u}$ and $\widetilde{\delta v}$. It is then indefinite on K_{t_f}, and so must be its restriction to Ker $F'_{t_c}(\overline{u})$. By virtue of Theorem 1.2, since $\overline{u}$ is a corank one critical point, F_{t_f} is locally open at $\overline{u}$ which prevents the point from being a local minimizer in $\mathrm{L}^\infty_m([0, t_f])$. ■

It is clear from the proof that in the analytic situation, if the first conjugate point t_{1c} belongs to $(0, t_f)$, the following happens. Under the Legendre assumption, all operators $B_{s|K_s}$ are Fredholm and so diagonalizable on a Hilbert basis of eigenvectors. For $s < t_{1c}$, all the eigenvalues are strictly negative. For t_{1c}, finitely many of them vanish (the kernel of a Fredholm operator is finite-dimensional [14]). For $s > t_{1c}$ at least one becomes positive, ensuring sign-indefiniteness. On the contrary, without analyticity, the kernel may remain nontrivial for $s > t_{1c}$ whith no positive eigenvalue appearing. The second order intrinsic derivative may so remain negative semi-definite on a nonempty interval of conjugate times.

In contrast with finite-dimensional optimization, one cannot obtain a sufficient condition for optimality by a compactness argument. However, there is a very satisfactory result asserting local optimality in a strong sense. Coming back to the initial control problem stated at the beginning of the section, we still assume that we have a regular extremal. Remembering that $\hat{x} = (x^0, x)$, Pontryagin maximum principle in strong form states that, if $\overline{u}$ is the bounded solution and $\overline{x}$ the associated Lipschitz trajectory, there exists a nonpositive constant p^0 and a Lipschitz function $\overline{p} : [0, t_f] \to (\mathbf{R}^n)^*$ such that almost everywhere on $[0, t_f]$,

$$\dot{x}(t) = \partial_p H(\overline{x}(t), \overline{u}(t), \overline{p}(t)), \quad \dot{p}(t) = -\partial_x H(\overline{x}(t), \overline{u}(t), \overline{p}(t)),$$

[1] In which case the manifolds X, U, and the data (f^0, f) have to be assumed smooth-analytic as well.

and

$$H(\overline{x}(t), \overline{u}(t), \overline{p}(t)) = \max_{u \in U} H(\overline{x}(t), u, \overline{p}(t)),$$

where (p^0 appears as a parameter)

$$H(x, u, p) = p^0 f^0(x, u) + p f(x, u).$$

If the *maximized Hamiltonian*, $(x, p) \mapsto \max_{u \in U} H(x, u, p)$, is well defined and smooth in a neighbourhood of $\overline{z}$, one necessarily has

$$H(x, p) = H(x, u(x, p), p) = \max_{u \in U} H(x, u, p)$$

on this neighbourhood under the Legendre condition (the implicit function $u(x, p)$ being locally the unique zero of $\partial_u H(x, \cdot, p)$, hence the only maximizer of $H(x, \cdot, p)$). These relations are homogenous in (p^0, p) and there are two cases: The abnormal case, $p^0 = 0$, and the normal one, $p_0 < 0$. A control is said to be $\mathscr{C}^0$-locally optimal whenever there exists a neighbourhood of the associated trajectory in $\mathscr{C}^0([0, t_f], \mathbf{R}^n)$ such that any other admissible trajectory[1] in this neighbourhood has a greater cost.

THEOREM 1.8 *For a normal regular extremal in the neighbourhood of which the maximized Hamiltonian is smooth, the absence of conjugate time on* $(0, t_f]$ *is sufficient for* $\mathscr{C}^0$*-local optimality.*

The proof is of completely different nature compared to the necessary condition. The argument is not spectral but relies on the geometric construction of a so-called *field of extremals* (see [1, 11]). Summarizing, under appropriate assumptions, before first conjugate time local optimality holds on a large $\mathscr{C}^0$-neighbourhood of the trajectory, while past t_{1c} local optimality is lost, even in small L^∞-neighbourhoods.

2. Differential pathfollowing

Assume we are given a family of optimal control problems parameterized by $\lambda \in [0, 1]$ in the form of §1. For each λ, we minimize

$$\int_0^{t_f(\lambda)} f^0(x(t), u(t), \lambda)\, \mathrm{d}t \to \min$$

with fixed final time $t_f(\lambda) > 0$, subject to differential constraints on the fixed manifold X,

$$\dot{x}(t) = f(x(t), u(t), \lambda), \quad u(t) \in U_\lambda, \quad t \in [0, t_f(\lambda)],$$

and boundary conditions

$$x(0) = x_0(\lambda), \quad x(t_f) = x_f(\lambda).$$

As before, we assume that manifolds U_λ are without boundary, and that all the data is smooth with respect to (x, u, λ) (*resp.* λ). We make the following uniform assumption:

[1] A trajectory with the fixed endpoints generated by a bounded control valued in U.

For all $\lambda \in [0,1]$, there exists a normal regular extremal $(x_\lambda, u_\lambda, p_\lambda)$ in the neighbourhood of which the maximized Hamiltonian is smooth, and along which Jacobi fields never become vertical for $t \in (0, t_f(\lambda)]$.

See [13] for a refined analysis in the Riemannian setting. Let $\lambda_0 \in [0,1]$. Under our assumption, the application

$$(p_0, \lambda) \mapsto x(t_f(\lambda), x_0(\lambda), p_0, \lambda)$$

mapping (p_0, λ) to the value at $t_f(\lambda)$ of the x-coordinate of the solution $z = (x, p)$ to

$$\dot{z}(t) = \overrightarrow{H}(z(t), \lambda), \quad t \in [0, t_f(\lambda)], \quad z(0) = (x_0(\lambda), p_0),$$

is a smooth implicit function in a neighbourhood of $(p_{\lambda_0}(0), \lambda_0)$. Herebefore, $H(x, p, \lambda) = H(x, u(x, p, \lambda), p, \lambda) = \max_{u \in U} H(x, u, p, \lambda)$ where $u(x, p, \lambda)$ is implicitly defined thanks to the Legendre condition. Optimal control *shooting* is a coordinate-dependent computation and we must choose a fixed chart on X whose domain contains the target points $x_f(\lambda)$ in order to define the homotopy function [20]. In such coordinates,

$$h(p_0, \lambda) = x(t_f(\lambda), x_0(\lambda), p_0, \lambda) - x_f(\lambda)$$

is smooth on the previous neighbourhood, possibly shrinked so $x_f(\lambda)$ stays in the chart. Finding the solution $p_{\lambda_0}(0)$ of the n-dimensional shooting equation $h(., \lambda_0) = 0$ provides us with an extremal along which no Jacobi field becomes vertical on $(0, t_f(\lambda_0)]$. In particular, there is no conjugate time on $(0, t_f(\lambda_0)]$ and we have a $\mathscr{C}^0$-locally optimal solution of optimal control problem with parameter λ_0. As target points $x_f(\lambda)$ must remain in the chart not only for λ close to λ_0 but for all $\lambda \in [0,1]$, we propose the following global framework.

Let $(x, \lambda) \mapsto \varphi(x, \lambda) \in \mathbf{R}^n$ be a parameterized chart defined on an open domain of the manifold (with boundary) $X \times [0,1]$. The graph $\{(x_f(\lambda), \lambda),\ \lambda \in [0,1]\}$ of x_f has to be a subset of this domain. Consider similarly the cotangent space $T^*_{x_0(\lambda)}$ above each point of the graph of x_0 to define the fiber space

$$\mathscr{F} = \bigcup_{\lambda \in [0,1]} \{x_0(\lambda)\} \times T^*_{x_0(\lambda)} \times \{\lambda\}$$

which is an $(n+1)$-submanifold (with boundary) of $T^*X \times [0,1]$. When the cotangent bundle is trivial, $T^*X \simeq X \times \mathbf{R}^n$, one simply has $\mathscr{F} = \mathbf{R}^n \times [0,1]$. The homotopy can then be globally defined on some open subset $\mathscr{F}_0$ of $\mathscr{F}$ to $\mathbf{R}^n$ putting

$$h(z, \lambda) = \varphi(x(t_f(\lambda), x_0(\lambda), p_0, \lambda), \lambda) - \varphi(x_f(\lambda), \lambda), \quad (z, \lambda) = (x_0(\lambda), p_0, \lambda) \in \mathscr{F}.$$

By restricting $\mathscr{F}_0$ if necessary, one can assume that it only contains regular points of h. As 0 is a regular value of the homotopy, each connected component of $h = 0$ is a 1-dimensional submanifold of $\mathscr{F}_0$, diffeomorphic either to $\mathbf{S}^1$ or to a real interval. A difficult issue is to provide sufficient conditions *à la Smale* [2] that ensure existence of a component joining $\lambda = 0$ to $\lambda = 1$. Other issues include the following, as illustrated in the trivial case $\mathscr{F}_0 = \Omega \times [0,1]$ with Ω an open subset of $\mathbf{R}^n$. (i) Components diffeomorphic to real intervals must have their endpoints in the topological boundary of $\Omega \times [0,1]$. Such endpoints are critical points of the homotopy (the path could be otherwise prolongated), and bifurcations may occur [2]. (ii) Besides, for a fixed λ, one has to compare the costs associated to the zeros in each component of $h(., \lambda) = 0$. This global aspect is responsible

for possible loss of regularity—even discontinuity—of the value function (mapping λ to the minimum value of the cost when it exists).

The connected component of the set of zeros starting at $(z_{\lambda=0}(0), 0)$ is a differentiable curve that we can parameterize by arc length,[1] s. In coordinates, $c(s) = (p_{\lambda(s)}(0), \lambda(s))$ is solution of the differential equation

$$\frac{\mathrm{d}c}{\mathrm{d}s}(s) = \overrightarrow{T}(c(s)), \quad c(0) = (p_{\lambda=0}(0), 0), \tag{4}$$

where $\overrightarrow{T}(c)$ is the unit tangent vector defined up to orientation by $\overrightarrow{T}(c) \in \operatorname{Ker} h'(c)$ (c is not critical so the kernel is one-dimensional). As critical points are avoided, orientation is choosen so that the sign of

$$\det\left[\ker h'(c)^T \;\; \overrightarrow{T}(c)\right]$$

remains constant. In contrast with established predictor-corrector methods ([31], *e.g.*), we follow the path of zeros by merely integrating the differential equation using a high order numerical scheme without making any correction step.

3. Algorithmic aspects

The numerical integration of (4) uses a high order one step scheme with variable stepsize. At each step, the tangent vector defining the right hand side has to be computed. This essentially amounts to evaluating the differential of the homotopy (and then to reveal its kernel by standard linear algebra). In coordinates,

$$\begin{aligned}
\partial_{p_0} h(p_0, \lambda) &= \partial_{p_0} x[p_0, \lambda], \\
\partial_\lambda h(p_0, \lambda) &= \partial_{t_f} x[p_0, \lambda] t_f'(\lambda) + \partial_{x_0} x[p_0, \lambda] x_0'(\lambda) + \partial_\lambda x[p_0, \lambda] - x_f'(\lambda),
\end{aligned}$$

where $[p_0, \lambda]$ stands for $(t_f(\lambda), x_0(\lambda), p_0, \lambda)$. The derivatives of t_f, x_0 and x_f are straightforwardly computed by AD. As $x[p_0, \lambda]$ is the x-component of the solution evaluated at t_f of the system defined by the maximized Hamiltonian,

$$\partial_{t_f} x[p_0, \lambda] = \nabla_p H(x[p_0, \lambda], p[p_0, \lambda])$$

where $p[p_0, \lambda]$ is the p-component of the same solution at t_f. For the variational derivatives $\partial_{x_0} x$, $\partial_{p_0} x$ and $\partial_\lambda x$, the first option is to use plain finite differences. Now, it is numerically crucial in applications to use variable stepsize methods to compute adapted discretizations thanks to local error control (nested Runge–Kutta methods do so at very low cost). In this case, finite differences are known to behave poorly as the two grids dynamically computed to evaluate, say, $x[p_0, \lambda]$ and $x[p_0 + \delta p_0, \lambda]$, may differ, generating an artificial nondifferentiability. The workaround is to force the grid to remain the same when evaluating for $p_0 + \delta p_0$. This is known as *Internal Numerical Derivative* (IND, [7]). An alternative is to use AD on the integration code. Because of the variable stepsize, the code only defines a piecewise differentiable function, and AD actually computes the piecewise derivative, providing the same order of precision as IND for explicit one-step variable stepsize methods [23]. The last option is to explicitly assemble the variational system to compute Jacobi fields. As $\overrightarrow{H}'$ is evaluated along the current solution, one has to integrate

[1] Under our assumption, the absence of conjugate point even implies that we can parameterize by λ, at least locally.

simultaneously z and δz systems ($2n + 4n^2$ dimensions to obtain the complete Jacobian with δz a matrix of order $2n$),

$$\dot{z}(t) = \overrightarrow{H}(z(t), \lambda), \quad \delta\dot{z}(t) = \partial_z \overrightarrow{H}(z(t), \lambda), \quad z(0) = z_0, \quad \delta z(0) = I_{2n}.$$

In the case of explicit one-step schemes with variable stepsize, AD on the integration code and VAR (integration of the augmented variational system) are identical provided the stepsize control in the latter case is made only on z-components (and not on $(z, \delta z)$). The same analysis holds for the variational derivative with respect to λ.

A byproduct of any of the three approaches, IND, AD or VAR, is the computation of Jacobi fields on $[0, t_f(\lambda)]$ which allows to verify numerically the assumption of §2. Let δz^i be the Jacobi fields with initial conditions $\delta z^i(0) = (0, e_i)$, $i = 1, n$, used to compute $\partial_{p_0} x[p_0, \lambda]$ ($\{e_1, \ldots, e_n\}$ being the canonical basis). The assumption that no Jacobi field along the extremal becomes vertical on $(0, t_f(\lambda)]$ is equivalent to check that the rank of $\{\delta x^i(t), \ldots, \delta x^n(t)\}$ is equal to n for $t \in (0, t_f(\lambda)]$. In practice, a sign change in the determinant of the Jacobi fields can be monitored, with a drawback on the magnitude as the determinant is equal to the product of the singular values, all vanishing for $t = 0$. A complementary test is thus to inspect the dynamics of the smallest singular value along the reference extremal.

The `hampath` compiler [15] implements these ideas. It extends `cotcot` [9, 10] by adding the homotopy layer as described in §2-3 and compiles a description via *Fortran* files of a regular family of optimal control problem into a collection of *Matlab* MEX-files. The entries are two files, `hfun.f` and `bcfun.f` defining respectively the smooth maximized Hamiltonian and the boundary conditions,

$$H(t, x, p, \lambda) : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}, \quad b(z_0, t_f, z_f, \lambda) : \mathbf{R}^{2n} \times \mathbf{R} \times \mathbf{R}^{2n} \times \mathbf{R} \to \mathbf{R}^{2n}.$$

The dynamics is not assumed to be autonomous and the final times may be left free, that is why t and t_f appear in the list of arguments of H and b, respectively. Boundary conditions are more general than what we have described and actually allow for initial and terminal constraints defined by proper submanifolds of the ambiant manifold rather than single points (again parameterized by λ),

$$x(0) \in X_{0,\lambda}, \quad x(t_f) \in X_{f,\lambda}.$$

Pontryagin maximum asserts that transversality conditions hold

$$p(0) \perp T_{x(0)} X_{0,\lambda}, \quad p(t_f) \perp T_{x(t_f)} X_{f,\lambda},$$

which, in coordinates, are translated into function b. In such a case, the absence of conjugate point along the reference extremal is still necessary but the condition is not sharp enough and one has to introduce *focal* points (see for instance [18] in the Riemannian case). The output of the compiler is a series of `Mex`-functions. Among these,

`expdhvfun` exponentiates $\partial_x \overrightarrow{H}$, that is integrates simultaneously the adjoint system in z an its linearized to compute Jacobi fields. The ODE integrator is `dopri5` [23] (explicit Runge–Kutta method of order 5 with dense output).

`ssolve` seeks a zero of the homotopy shooting function for a single λ. Typically, it is used to compute the initial condition for pathfollowing. The Newton solver is `hybrj` [19] (modified Powell hybrid method). The Jacobian of the shooting function is computed by the VAR approach.

`hampath` follows the path of zero by merely integrating differential equation (4).

As is clear from Fig. 1 overview of the code organisation, several first and second order derivatives have to be generated. This is done by AD using TAPENADE [24]. In addition to the derivatives listed at the beginning of the section, the symplectic gradient and its derivative (that is the Hessian of H up to some block permutation) are computed by AD, then is used to assemble the adjoint system and the Jacobi one as well.

4. Example 1: Quantum control

We address the energy minimization problem for two-level dissipative quantum systems [8, 12]. It consists in minimizing the L^2-norm of the control over a fixed transfer time t_f, while steering an initial state q_0 to a final state q_1. The state is $q = (x, y, z) \in \mathbf{R}^3$ and the dynamics is given by the control system

$$\begin{aligned}
\dot{x} &= \Gamma x + u_2 z,\\
\dot{y} &= -\Gamma y - u_1 z,\\
\dot{z} &= \gamma_- - \gamma_+ + u_1 y - u_2 x.
\end{aligned}$$

This system is deduced from Kossakowski–Lindblad equations describing the dynamics of two-level dissipative quantum systems in the *Rotating Wave Approximation*. Up to a proper renormalization, it corresponds to the Bloch equation in Nuclear Magnetic Resonance (NMR), when the detuning term is zero. The dissipative parameters satisfy the constraint $2\Gamma \geq \gamma_+ \geq |\gamma_-|$, and the control is the complex function $u = u_1 + iu_2$ modelizing the action of an electromagnetic field. The cost corresponds to the energy transfer between the control field and the internal Hamiltonian and is an important physical issue. The Bloch ball $|q| \leq 1$, which is the physical state space of the system, is invariant for the dynamics. Such a control problem is motivated by two recent experimental research projects. The first one concerns the control of the rotation of a molecule in a gas phase by using laser fields, and the second deals with the control of the spin dynamics by a magnetic field in NMR.

If we use spherical coordinates $x = \rho \sin\phi \cos\theta, y = \rho \sin\varphi \sin\theta\,, z = \rho\cos\theta$, the state equation becomes

$$\begin{aligned}
\dot{\rho} &= \gamma_- \cos\varphi - \rho(\delta \cos^2\varphi + \Gamma),\\
\dot{\theta} &= -v_1 \cot\varphi,\\
\dot{\varphi} &= -\frac{\gamma_- \sin\varphi}{\rho} + \delta \sin\varphi\cos\varphi + v_2,
\end{aligned}$$

where $\delta = \gamma_+ - \Gamma$ and the new control is $v = e^{i\theta}u = v_1 + iv_2$. Note in particular that the cost is invariant by this feedback as

$$\int_0^{t_f} (v_1^2 + v_2^2)\,\mathrm{d}t = \int_0^{t_f} (u_1^2 + u_2^2)\,\mathrm{d}t.$$

The initial and final state are here $(\rho(0), \theta(0), \varphi(0)) = (1, 0, \pi/4)$ and $(\rho(t_f), \theta(t_f), \varphi(t_f)) = 0.3534, 2.053, 0.6004)$. The final time is set to $t_f = 0.5$. Parameters γ_+ and γ_- are fixed respectively to 2 and 0.1, and we are interested here by the evolution of the solution when the parameter Γ is decreased from 3 to 2. The corresponding path of zeros is portrayed Fig. 2. Optimal controls and trajectories for intermediate Γ are represented Figs. 3 and 4. Sufficient second order conditions are verified by a rank test on Jacobi fields along these

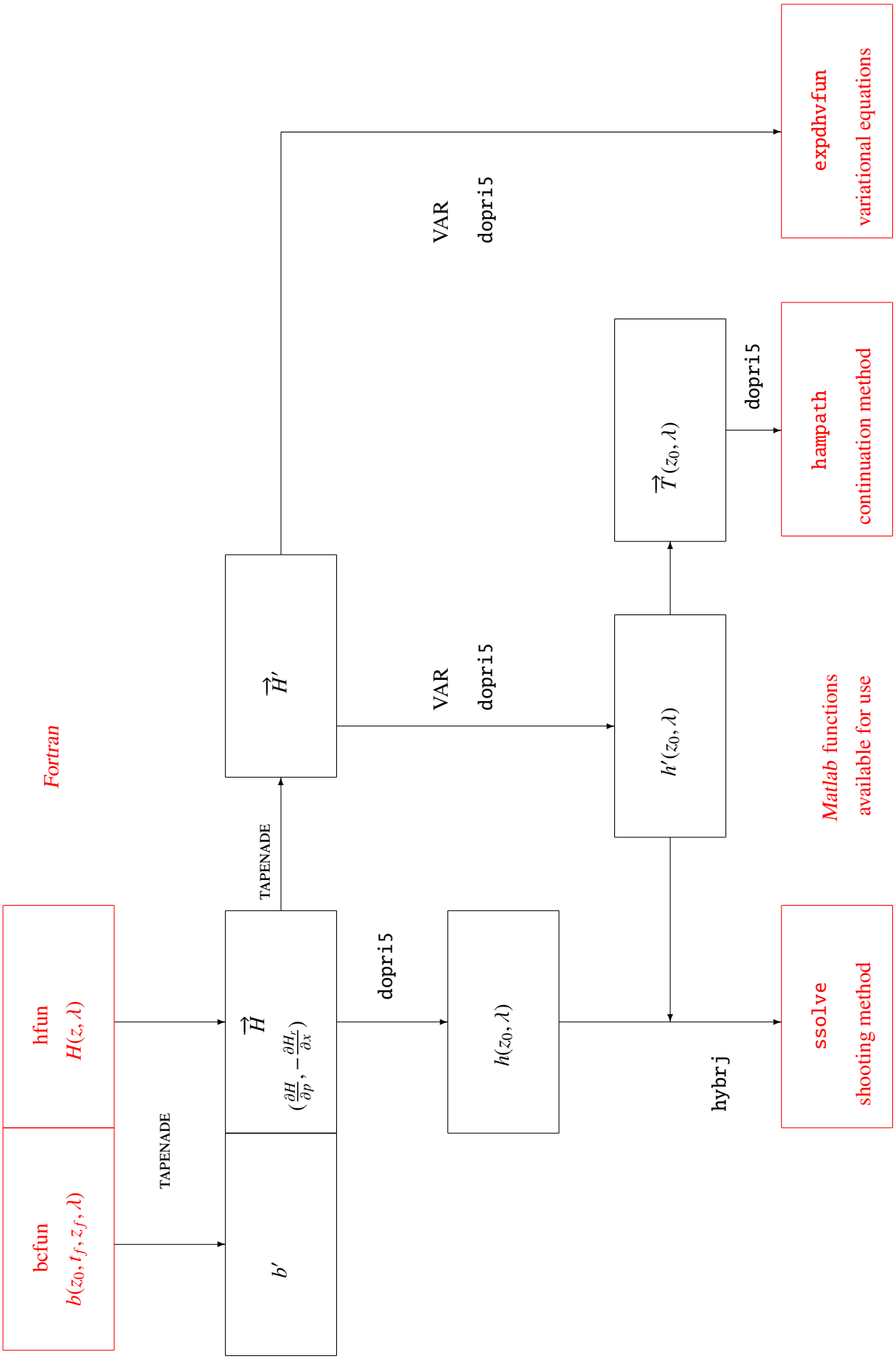

Figure 1. Overview of `hampath`.

extremals, see Fig. 5, ensuring that we obtain $\mathscr{C}^0$-local optimal solutions of the problem in the whole range of physical parameters.

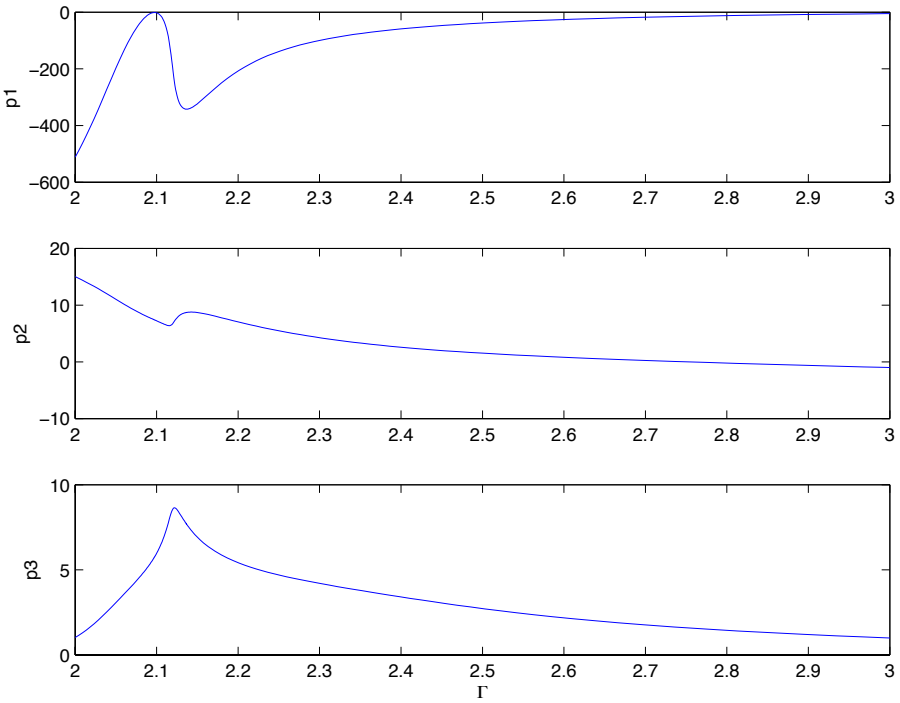

Figure 2. Energy minimization in quantum control, path of zeros. The initial adjoint state $p_0 \in (\mathbf{R}^3)^*$ solution of the shooting function is computed when the homotopic parameter Γ decreases from 3 to 2.

5. Example 2: Two-body control

Motivated by orbit transfer problems in space mechanics [3, 27], we study the two-body controlled equation,

$$\ddot{q}(t) = -\mu \frac{q(t)}{|q(t)|^3} + \frac{\varepsilon u(t)}{m}, \quad |u| \leq 1.$$

The position vector q is in $\mathbf{R}^3$ and we restrict to non-collision elliptic orbits,

$$q \times \dot{q} \neq 0, \quad \dot{q}^2 < \frac{2\mu}{|q|},$$

which defines the six-dimensional state manifold here in the Cartesian coordinates. The control u is prescribed to the unit Euclidean ball, $u_1^2 + u_2^2 + u_3^2 \leq 1$, and $\varepsilon u/m$ where m is the mass of the second body models the acceleration at our disposal. Typically, εu is the thrust of a spacecraft in the central field of a planet (the first body) whose gravitation constant is μ. For small ε, the trajectory is a perturbation of the integrable Keplerian motion. Dimension five coordinates associated to first integrals that describe the geometry of the osculating ellipse, plus an angular position parameter, are well suited for numerical computations as all but the last one will be slowly varied during the motion. The *semi-latus rectum* P and the *eccentricity* e prescribe the shape of the ellipse. The three Euler angles, Ω (precession angle or *ascending node longitude*), i (nutation angle or *inclination*), θ (proper rotation or *argument of perigee*), prescribe the rotation of the orbit plane treated

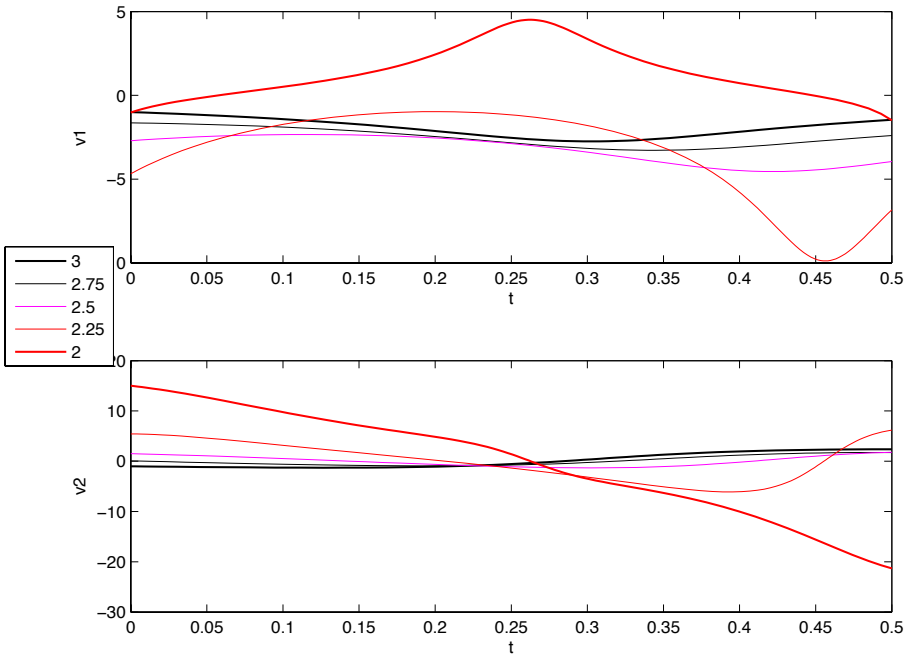

Figure 3. Energy minimization in quantum control, controls. The two components of optimal controls for $\Gamma = 3$, 2.75, 2.5, 2.25 and 2 are portrayed, illustrating the change of strategy when the physical parameter is decreased.

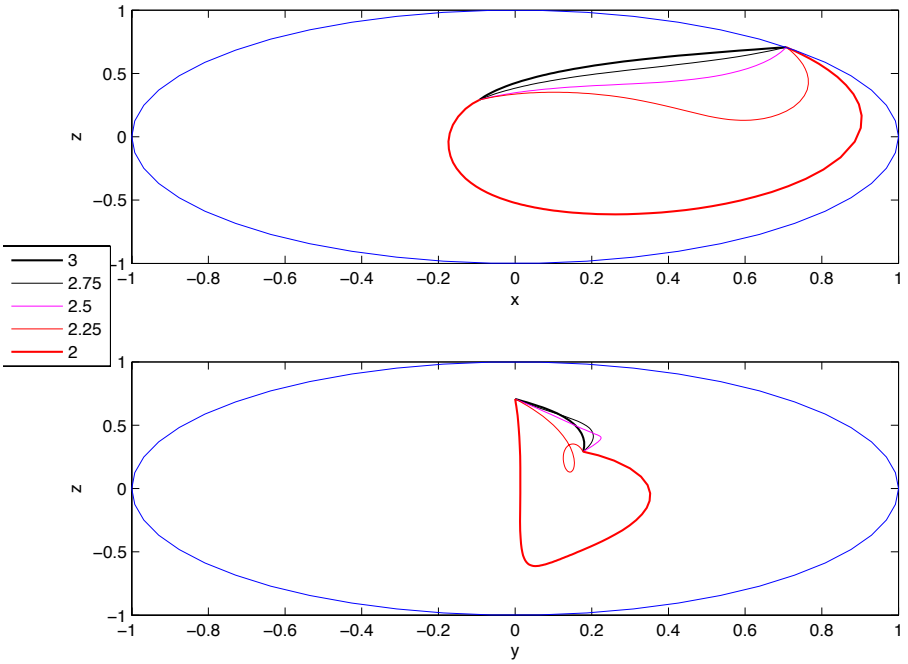

Figure 4. Energy minimization in quantum control, trajectories. Optimal trajectories in the Bloch ball are projected in the (x, z)-plane (upper subplot) and (y, z)-plane (lower subplot) for $\Gamma = 3$, 2.75, 2.5, 2.25 and 2.

as a rigid body. Adding *longitude*, l, to define the position of the ellipse itself ($l - (\Omega + \theta)$ measures the polar angle with respect to semi-major axis in the orbit plane), we obtain a complete set of coordinates. Equivalently, we choose $(P, e_x, e_y, h_x, h_y, l)$ where the two vectors

$$(e_x, e_y) = e(\cos(\Omega + \theta), \sin(\Omega + \theta)), \quad (h_x, h_y) = \tan(i/2)(\cos\Omega, \sin\Omega),$$

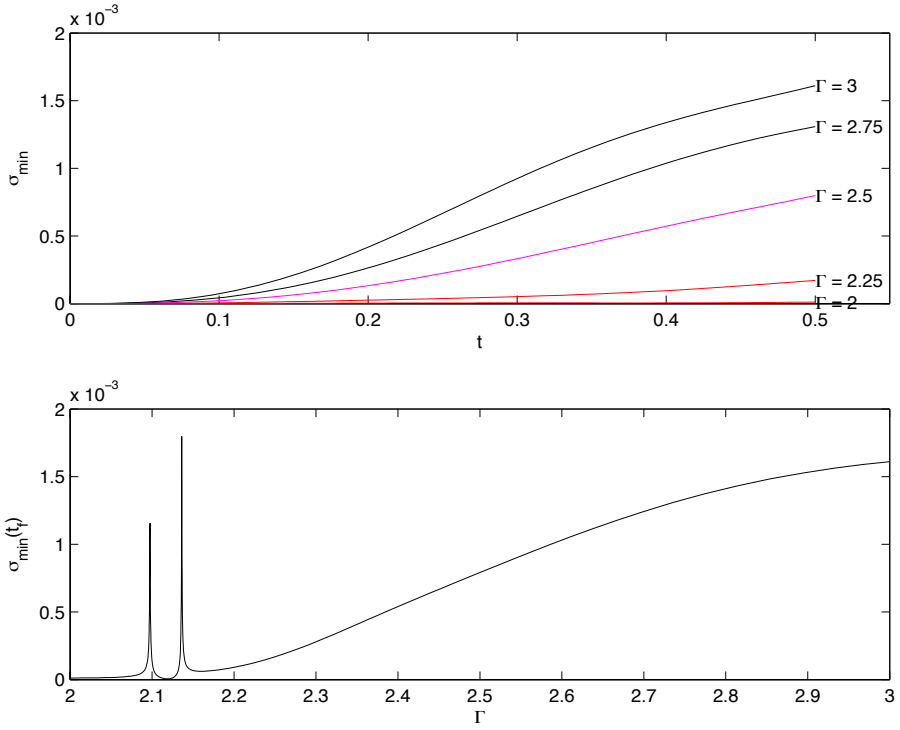

Figure 5. Energy minimization in quantum control, second order condition check. Along each extremal for $\Gamma = 3$, 2.75, 2.5, 2.25 and 2, three Jacobi fields are computed and the rank of $\{\delta x^1(t), \delta x^2(t), \delta x^3(t)\}$ is evaluated by SVD. The evolution of the smallest singular value σ_3 is portrayed in each case (upper subplot). A careful check shows that it remains positive on $(0, t_f]$ (even in the neighbourhood of $\Gamma = 2.1$), so second order sufficient conditions apply and settle $\mathscr{C}^0$-local optimality. The evolution of $\sigma_3(t_f)$ along the path, directly connected to the rank condition of the homotopy, is also given (lower subplot).

are oriented by the semi-major axis (*eccentricity vector*) and the line of nodes, respectively. In these coordinates, the dynamics reads

$$\dot{x}(t) = F_0(x(t)) + \frac{\varepsilon}{m} \sum_{i=1}^{3} u_i(t) F_i(x(t))$$

with

$$F_0(x) = \sqrt{\frac{\mu}{P}} \frac{W^2}{P} \frac{\partial}{\partial l},$$

$$F_1(x) = \sqrt{\frac{P}{\mu}} \left(\sin l \frac{\partial}{\partial e_x} - \cos l \frac{\partial}{\partial e_y} \right),$$

$$F_2(x) = \sqrt{\frac{P}{\mu}} \left(\frac{2P}{W} \frac{\partial}{\partial P} + (\cos l + \frac{e_x + \cos l}{W}) \frac{\partial}{\partial e_x} + (\sin l + \frac{e_y + \sin l}{W}) \frac{\partial}{\partial e_y} \right),$$

$$F_3(x) = \sqrt{\frac{P}{\mu}} \left(-\frac{Ze_y}{W} \frac{\partial}{\partial e_x} + \frac{Ze_x}{W} \frac{\partial}{\partial e_y} + \frac{C \cos l}{2W} \frac{\partial}{\partial h_x} + \frac{C \sin l}{2W} \frac{\partial}{\partial h_y} \right),$$

and

$$W = 1 + e_x \cos l + e_y \sin l, \quad Z = h_x \sin l - h_y \cos l, \quad C = 1 + h_x^2 + h_y^2.$$

In contrast with §4, the cost functional is the L^1-norm of the control,

$$\int_0^{t_f} |u(t)|\,\mathrm{d}t \to \min, \quad |u(t)| = \sqrt{u_1(t)^2 + u_2(t)^2 + u_3(t)^2},$$

with free final time t_f. Such a performance index models the fuel consumption (here with the approximation that the mass, m, remains constant) and is much more complicated than an L^2-cost as is clear when applying Pontryagin maximum principle.

The normal Hamiltonian to be maximized (the abnormal case is trivially eliminated here, so we set $p^0 = -1$) is

$$H(x, u, p) = -|u| + H_0(x, p) + \frac{\varepsilon}{m} \sum_{i=1}^{3} u_i H_i(x, p)$$

where each $H_i(x, p) = pF_i(x)$ is the Hamiltonian lift of the corresponding vector field. Let us define the two switching functions

$$\psi(x, p) = (H_1, H_2, H_3)(x, p), \quad \varphi(x, p) = -1 + \frac{\varepsilon}{m}\psi(x, p).$$

On the open dense subset $\psi \neq 0$, the maximizer of $H(x, \cdot, p)$ on the unit Euclidean ball is

$$u = \frac{\psi}{|\psi|} \quad \text{when} \quad \varphi > 0, \quad u = 0 \quad \text{when} \quad \varphi < 0,$$

arbitrary if φ vanishes. As a result, the control norm is bang-bang (equal to 0 or 1), and the problem is very difficult to solve without *a priori* knowledge on the switching structure [26]. We therefore propose a standard interior penalization using a logarithmic barrier [4, 5] to eliminate simultaneously the lack of regularity coming from the nonemptyness of the control set boundary ($|u| = 1$) and from the non-differentiability of the cost integrand at $u = 0$. For $\alpha \in [0, 1]$, consider the cost

$$\int_0^{t_f} [|u| - (1 - \alpha) \ln |u| - (1 - \alpha) \ln(1 - |u|)]\,\mathrm{d}t$$

for u in the open pointed unit ball, $0 < |u| < 1$. Moreover, we combine this approach with a second perturbation putting

$$\int_0^{t_f} \{(1 - \beta) + \beta[|u| - (1 - \alpha) \ln |u| - (1 - \alpha) \ln(1 - |u|)]\}\,\mathrm{d}t.$$

When β is zero, the cost is t_f and we have the minimum time problem which is much simpler and well studied [16]. For $\beta = 1$ and $\alpha = 1$ the original problem is retrieved, while for $\beta \in [0, 1]$ and $\alpha < 1$, we are in the regular situation described in §1-2. Another approach would consist in solving directly the target problem. Although second order conditions that treat mixed control-state constraints are available [25], this would have two disadvantages in our view. First, one has to guess the number and location of junction points with the boundary of the constraint (whereas following the path turns numerically to capture the structure as illustrated by Fig. 7). Moreover, these junction points set up new unknowns of the problem, making its solution even costlier (for low thrusts, that is for small ε, there are hundreds of junctions). Rather than cascading the two homotopies, we use a single continuation $\lambda \mapsto (\alpha(\lambda), \beta(\lambda))$. The question of convergence when $\alpha \to 1$

is an issue though some hints can be found in the literature (see, *e.g.*, [29, 30]). See also [21, 22] for results on this problem using another homotopy.

The results below solve a problem originally posed by the French Space Agency. A satellite of mass $m = 1500$ Kg around the Earth ($\mu = 398600.47$ km^{3}s^{-2}) is to be transferred from an initial low, eccentric and inclined orbit towards the geostationary one using an $\varepsilon = 10$ Newtons thrust. In our chart,

$$\begin{array}{llllll} P_0 = 11625 \text{ Km}, & e_{x0} = 0.75, & e_{y0} = 0, & h_{x0} = 0.612, & h_{y0} = 0, & l_0 = \pi \text{ rad}, \\ P_f = 42165 \text{ Km}, & e_{xf} = 0, & e_{yf} = 0, & h_{xf} = 0, & h_{yf} = 0, & l_f = 56 \text{ rad}. \end{array}$$

Numerical simulations are displayed for λ between 0 and $1 - 1e - 3$. The corresponding path of zeros is portrayed Fig. 6. Optimal controls for intermediate λ (close to 0, or close to 1) are represented Fig. 7. The trajectory for $\lambda = 0.999$ is at Fig. 8. Sufficient second order conditions are verified by a rank test on Jacobi fields along these extremals, see Fig. 9, ensuring that we obtain $\mathscr{C}^0$-local optimal solutions of the problem for $\lambda < 1$.

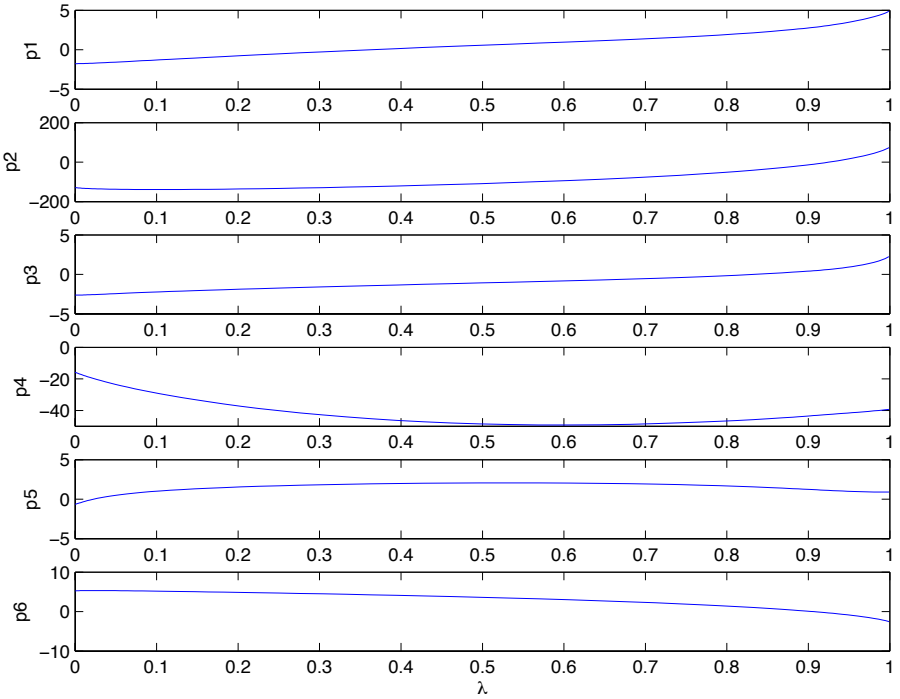

Figure 6. L^1-minimization in two-body control, path of zeros. The initial adjoint state $p_0 \in (\mathbf{R}^6)^*$ solution of the shooting function is computed when the homotopic parameter tends to 1.

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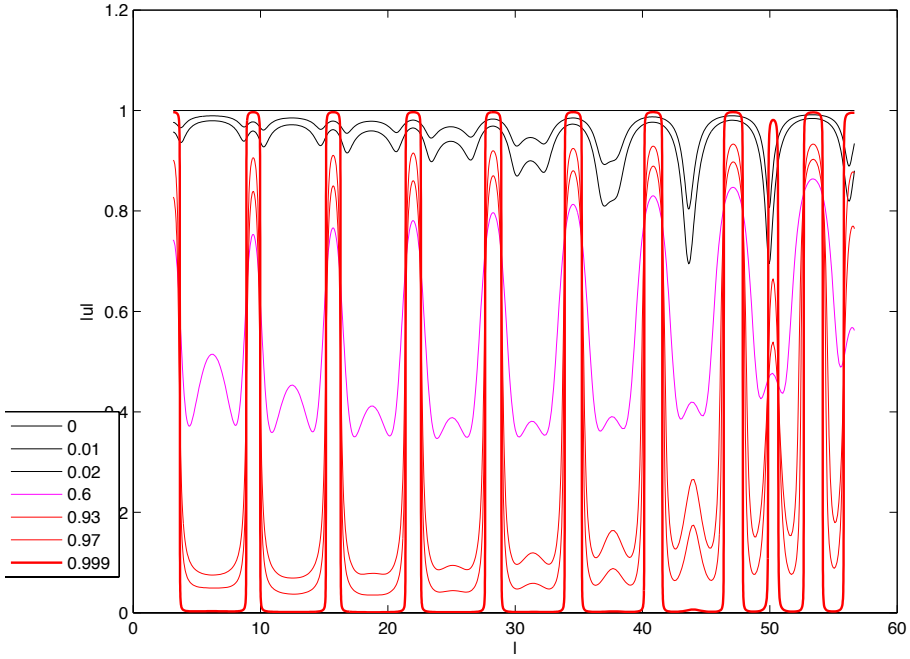

Figure 7. L^1-minimization in two-body control, norm of the controls. The norm of the three-dimensional control versus longitude (the final longitude is fixed whereas the final time is free) is displayed for λ close to 0, $\lambda = 0.6$, and λ close to 1. For $\lambda = 0$, the minimum time problem is solved and $|u| = 1$ everywhere [16]. Conversely, for λ close to 1, the switching structure has been captured and almost-switches between 0 and 1 are observed on the norm.

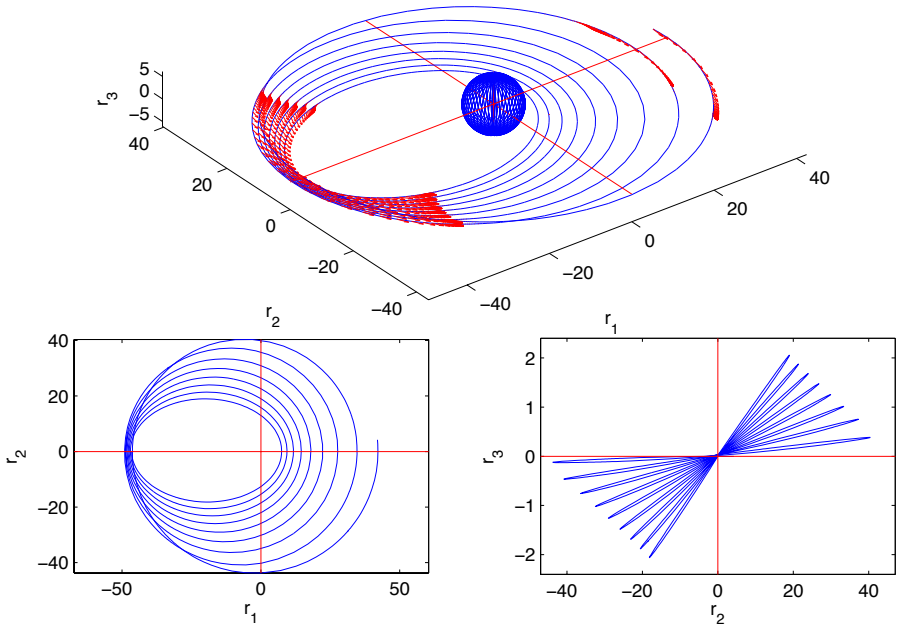

Figure 8. L^1-minimization in two-body control, optimal trajectory. The trajectory (in blue) around the Earth is displayed for $\lambda = 0.999$, in three dimensions (upper subplot), (q_1, q_2) and (q_2, q_3)-projections (lower subplots). The red arrows indicate the control in the three-dimensional view. The action of the control is clearly located around the apogees and the last two perigees.

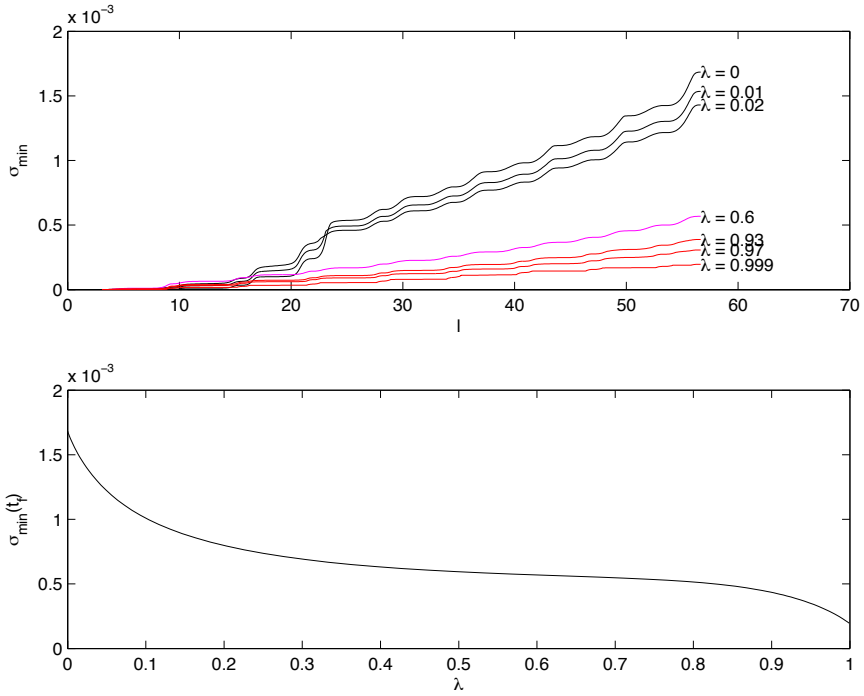

Figure 9. L^1-minimization in two-body control, second order condition check. Along each extremal for the previous values of λ, six Jacobi fields are computed and the rank of $\{\delta x^1(t), \ldots, \delta x^6(t)\}$ is evaluated by SVD. The evolution of the smallest singular value σ_6 over longitude is portrayed in each case (upper subplot). As it remains positive on $(0, t_f(\lambda)]$, second order sufficient conditions apply and settle $\mathscr{C}^0$-local optimality. The evolution of $\sigma_6(t_f)$ along the path, directly connected to the rank condition of the homotopy, is also given (lower subplot).

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