Hampath – on solving optimal control problems by indirect and path following methods

STRUCTURAL DYNAMICAL SYSTEMS : Computational Aspects
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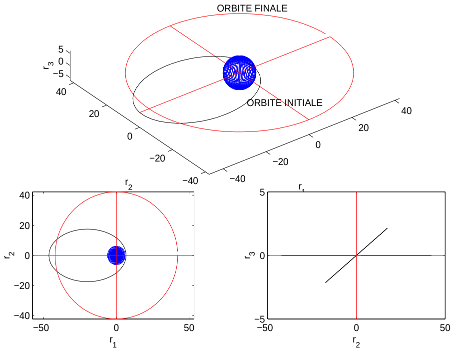
Motivation



contract

- LEO - GOE transfer : $P_0 = 11.625$ Mm, $e_0 = 0.75$ et $i_0 = 7°$
- Initial mass : $m_0 = 1500$ kg
- Low thrust : $T_{\max} = 0.1$ N
- Costs
  - Minimization of t_f
  - Maximization of the final mass
  - Minimization of the “energy”

Equations and results

$$
(P)\begin{cases}
\min \int_0^{t_f} |u|dt \iff \max m(t_f) \\[1em]
\begin{aligned}
\dot{r} &= v \\
\dot{v} &= -\frac{\mu_0}{|r|^3} r + T_{\max}\frac{u}{m} \\
\dot{m} &= -\beta T_{\max}|u|
\end{aligned} \\[1em]
|u| \leq 1 \\[1em]
\text{Initial and final conditions} \\
t_f \quad \text{free}
\end{cases}
$$

$|u|$: euclidian norm

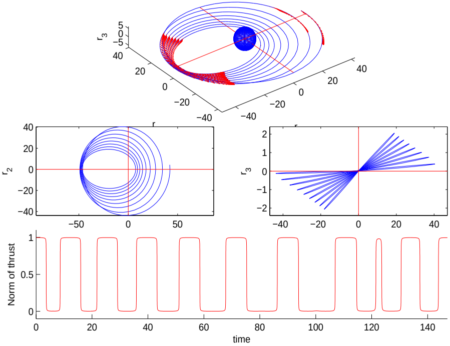


FIGURE: Optimal trajectory and norm of the optimal control for $T_{\max} = 10\text{N}$

Issues and difficulties

Issue The control u is discontinuous (bang-bang solution)

$\Rightarrow$ (BVP) with discontinuities

Idea Regularize and converge to the problem.

Problems solved

$$
(P_\lambda)\left\{
\begin{array}{l}
Min \quad g(x(t_f),\lambda) + \int_0^{t_f} f^0(x(t),u(t),\lambda)dt \\
\\
\dot{x}(t) = f(x(t),u(t),\lambda) \quad , \quad u(t) \in U \subset \mathbf{R}^m \quad , \quad \lambda \in \mathbf{R} \\
\\
b_0(x(0),\lambda) = 0 \in \mathbf{R}^{n_0} \quad , \quad n_0 \leq n \\
b_f(x(t_f),\lambda) = 0 \in \mathbf{R}^{n_1} \quad , \quad n_1 \leq n
\end{array}
\right.
$$

Definition

The Hamiltonian function is

$$
\begin{array}{rcl}
H : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R} \times \mathbf{R}^- \times \mathbf{R}^n & \longrightarrow & \mathbf{R} \\
(x,u,\lambda,p^0,p) & \longmapsto & H(x,u,\lambda,p^0,p)
\end{array}
$$

$$
H(x,u,\lambda,p^0,p) = p^0 f^0(x,u,\lambda) + < p, f(x,u,\lambda) >
$$

Hypothesis

- Normal case : $p^0 \neq 0 \Rightarrow p^0 = -1$
- All functions are smooth
- The maximization of the Hamiltonian gives a smooth function $u(x,p,\lambda) = argmax_{w \in U} H(x,w,\lambda,p)$

Definition

We call true Hamiltonian the function

$$H_r(x,\lambda,p) = H(x,u(x,p,\lambda),\lambda,p)$$

Example

$$(P_\lambda)\begin{cases}\min \int_0^1 u^2 dt\\ \dot{x} = -\lambda x + u\\ x(0) = -1\\ x(1) = 0\end{cases}$$

The Hamiltonian is $H(x, u, \lambda, p) = -u^2 + p(-\lambda x + u)$,

The maximization of the Hamiltonian gives $u(x, p, \lambda) = p/2$,

the true Hamiltonian is $H_r(x, \lambda, p) = \frac{p^2}{4} - \lambda xp$

and the *Principle of the Maximum of Pontryagin* (PMP) does not give conditions on $p(0)$ nor $p(1)$ (because $x(0)$ and $x(1)$ are known).

Shooting method

$$(P_\lambda)$$

↓ (PMP)

$$(BVP_\lambda)\begin{cases} \dot{x} = -\lambda x + \frac{p}{2} \\ \dot{p} = \lambda p \\ x(0) + 1 = 0 \\ x(1) = 0 \end{cases}$$

↓ (Shooting method)

$$S(x(0), p(0), \lambda) = \begin{pmatrix} x(0) + 1 \\ x(1) \end{pmatrix} = 0$$

with $x(1)$ obtained from integration (it depends on $x(0)$, $p(0)$ and λ).

Generally

$$(P_\lambda)$$

$\downarrow$ (PMP)

$$(BVP_\lambda)\begin{cases} (\dot{x},\dot{p}) = \vec{H}(x,\lambda,p) = \left(\frac{\partial H_r(x,\lambda,p)}{\partial p}, -\frac{\partial H_r(x,\lambda,p)}{\partial x}\right) \\ b_0(x(0),p(0),\lambda) = 0 \in \mathbf{R}^n \\ b_f(x(t_f),p(t_f),\lambda) = 0 \in \mathbf{R}^n \end{cases}$$

$\downarrow$ (Shooting method)

$$S(x_0,p_0,\lambda) = \begin{pmatrix} b_0(x_0,p_0,\lambda) \\ b_f(x_f,p_f,\lambda) \end{pmatrix} = 0 \qquad (S : \mathbf{R}^{2n+1} \to \mathbf{R}^{2n})$$

with $(x_f,p_f) \equiv (x(t_f,x_0,p_0,\lambda),p(t_f,x_0,p_0,\lambda)) = exp_{t_f}\vec{H}(x_0,\lambda,p_0)$

Introduction to the homotopy

We want to compute the zeros path of the homotopic function

$$\begin{array}{rcl} S : \mathbf{R}^{2n} \times [0,1] & \longrightarrow & \mathbf{R}^{2n} \\ (z_0, \lambda) & \longmapsto & S(z_0, \lambda) \end{array}$$

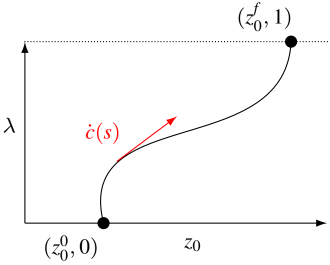
S is nonlinear and smooth. Assuming that 0 is a regular value for S, the solution set of $S(c) = 0$ is a smooth curve.
Path following techniques :

- Prediction-Correction (Allgower and Georg. [2])
- Path following by SVD (Dieci and al. [8])
- **hampath** for optimal control [4]

Tangent vector computation

Assuming that $c(s) = (z_0(s), \lambda(s))$ is such as

1. $c(0) = (z_0^0, 0)$
2. $S(c(s)) = 0$
3. $S'(c(s))$ of full rank
4. $\dot{c}(s) \neq 0$



then $\dot{c}(s) = T(S'(c(s)))$ is determined by

1. $S'(c(s))\dot{c}(s) = 0$
2. $|\dot{c}(s)|$=1
3. $\det \begin{pmatrix} S'(c(s)) \\ {}^t\dot{c}(s) \end{pmatrix}$ is of constant sign

Differential algorithms

- PC [2] : an Euler step for the prediction and a Newton method for the correction.

- By smooth SVD [8] : smooth update of the Jacobian decomposition factors.

- **hampath** [4] uses DOPRI5 from E. Hairer and G. Wanner [6] [7], for the numerical integration of (without any correction) :

$$(IVP)\left\{\begin{array}{l} \dot{c}(s) = T(S'(c(s))) \\ c(0) = (z_0^0, 0) \end{array}\right.$$

Until s_f such as $\lambda(s_f) = 1$ (dense output).

$$S(z_0, \lambda) = bc(z_0, z(t_f, z_0, \lambda), \lambda)$$

with

$$bc(z_0, z_f, \lambda) = (b_0(z_0, \lambda)\ ,\ b_f(z_f, \lambda))$$

$$\Rightarrow\ S'(z_0, \lambda) = \left[\frac{\partial bc}{\partial z_0} + \frac{\partial bc}{\partial z_f}\frac{\partial z}{\partial z_0} \quad \frac{\partial bc}{\partial \lambda} + \frac{\partial bc}{\partial z_f}\frac{\partial z}{\partial \lambda} \right]$$

- **hampath** uses TAPENADE [5] (from INRIA) for the automatic differentiation : $\frac{\partial bc}{\partial z_0}$, $\frac{\partial bc}{\partial z_f}$ and $\frac{\partial bc}{\partial \lambda}$ computation.
- **hampath** generates the *variational equations* to compute $\frac{\partial z}{\partial z_0}$ and $\frac{\partial z}{\partial \lambda}$.

$S'(z_0, \lambda)$

- $\frac{\partial z}{\partial z_0}(t_f, z_0, \lambda)$ is solution of :

$$(VAR) \left\{ \begin{array}{l} \dot{Y}(t) = \frac{\partial \vec{H}}{\partial z}(z(t, z_0, \lambda), \lambda) Y(t) \\ Y(0) = I_{2n} \end{array} \right.$$

  with $\frac{\partial \vec{H}}{\partial z}$ and $\vec{H} = (\frac{\partial H_r}{\partial p}, -\frac{\partial H_r}{\partial x})$ computed by automatic differentiation (TAPENADE).

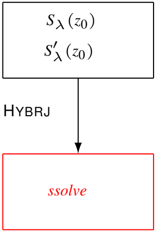
  Recall : $z = (x, p)$ and H_r is the true Hamiltonian.

- Same thing for $\frac{\partial z}{\partial \lambda}(t_f, z_0, \lambda)$ and $\frac{\partial \vec{H}}{\partial \lambda}$.

ssolve and expdhvfun methods

$$\begin{cases} \dot{Y}(t) = \frac{\partial \vec{H}}{\partial z} Y(t) \\ Y(0) = Y_0 \end{cases}$$

expdhvfun integrates the variational equations, with any initial condition. It is used to check the second order optimality conditions (cf [3] et [1]).



ssolve uses HYBRJ (Minpack library) as NLE solver.

Besides **hampath** integrates the variational equations so that this diagram commutes (DOPRI5 has been modified) :

$$\begin{array}{ccc} (IVP) & \xrightarrow{\text{Numerical integration}} & S(z_0, \lambda) \\ \text{Derivative} \downarrow & & \downarrow \text{Derivative} \\ (VAR) & \xrightarrow{\text{Numerical integration}} & \frac{\partial S}{\partial z_0}(z_0, \lambda) \end{array}$$

hampath use

User implementation (example)

$bc(z_0, z_f, \lambda)$

```
Subroutine bcfun(t0,tf,n,z0,zf,hf,lpar,par,nbc,bc)
  !Variable declarations
  bc = (/ z0(1)+1 , zf(1)/)
end subroutine bcfun
```

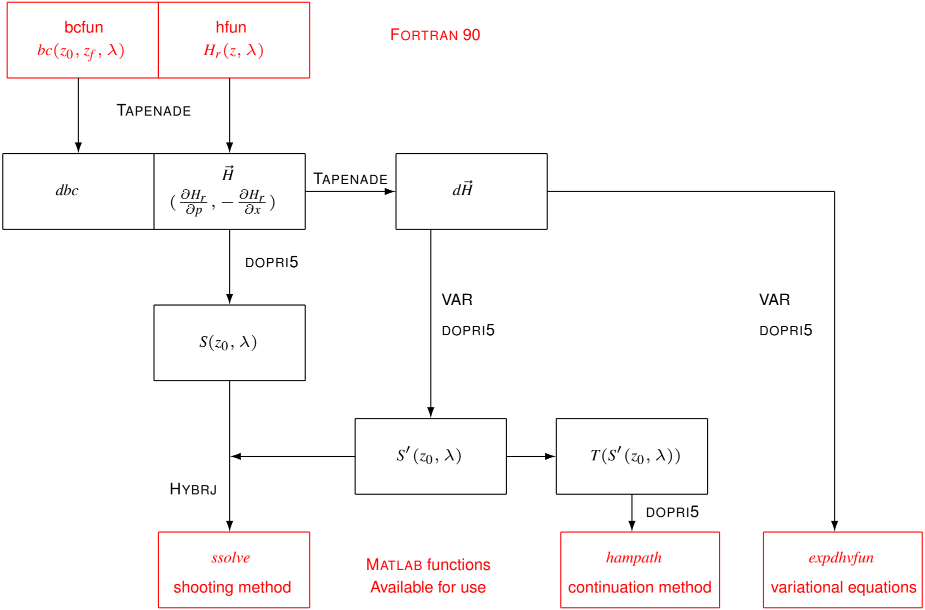
$H_r(z, \lambda)$

```
Subroutine hfun(t,n,z,lpar,par,H)
 !Variable declarations
  x = z(1)
  p = z(2)
  lambda = par(1)
  H = p**2/4 - lambda * x * p
end subroutine hfun
```

Available MATLAB functions after compiling

ssolve to solve the problem at λ fixed ;
hampath to follow the zeros path ;
expdhvfun to check the optimality of the solution.

Global diagram



Homotopy chosen

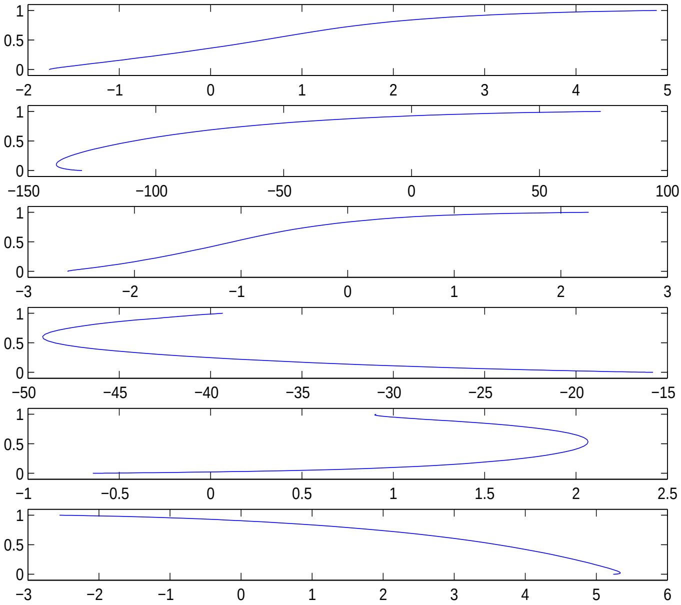
$$Min \quad \int_0^{t_f} |u| dt$$

$$\Downarrow$$

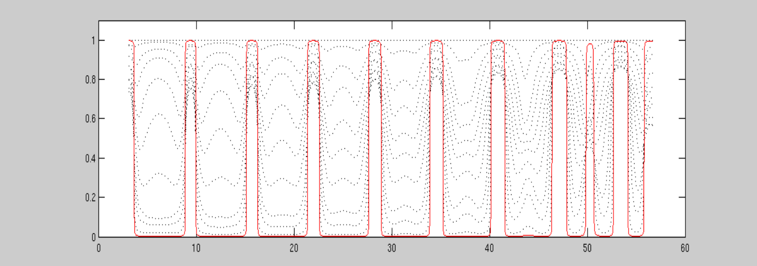
$$Min \ (1-\lambda)t_f + \lambda \int_0^{t_f} |u| - (1-\lambda)\left(\log|u| + \log(1-|u|)\right) dt$$

- For $\lambda = 0$ the final time t_f is minimized ;
- For $\lambda = 1$ the final mass is maximized ;
- For $\lambda \in [0, 1)$ the control $u(x, p, \lambda)$ is smooth.

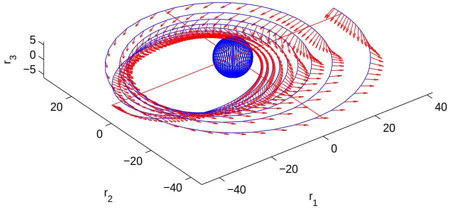
Zeros path - λ vs costate



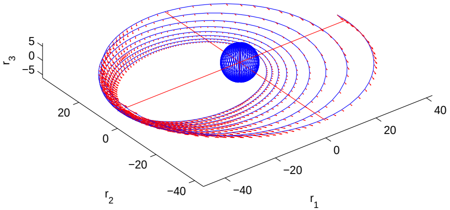
The control



$|u|$ for $\lambda \in \{0, 0.01, 0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.99, 0.999\}$

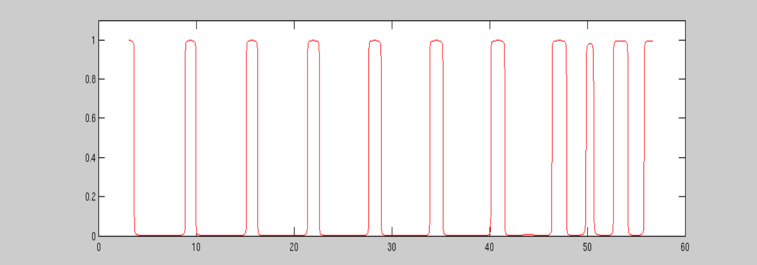


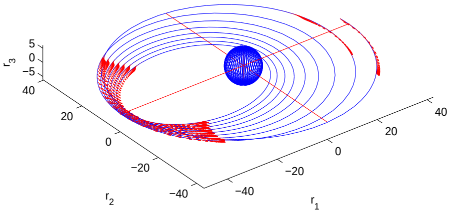
$\lambda = 0$



$\lambda = 0.9$

The control





$\lambda = 0.999$

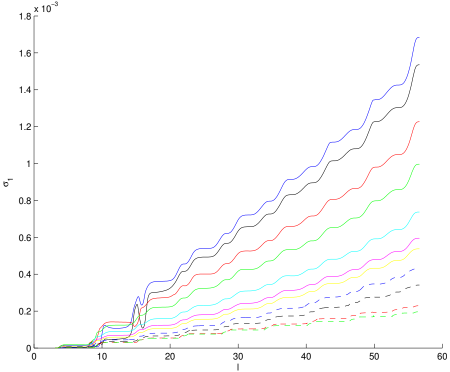
Locally optimality (assuming t_f fixed)

$$
\begin{array}{rcl}
exp_{t,x_0,\lambda} : \mathbf{R}^n & \longrightarrow & \mathbf{R}^n \\
p_0 & \longmapsto & x(t, x_0, p_0, \lambda)
\end{array}
$$

(x_0, p_0) is a locally optimal solution

$\Leftrightarrow \forall t \in (0, t_f]\ rank(\frac{d}{dp_0} exp_{t,x_0,\lambda}(p_0)) = n$

$\Leftrightarrow \forall t \in (0, t_f]\ \frac{\partial x}{\partial p_0}(t, x_0, p_0, \lambda) \in \mathbf{R}^{n \times n}$ is of full rank



Smallest singular value w.r.t. L

for $\lambda \in \{0, 0.01, 0.05, 0.1, 0.25, 0.5, 0.75, 0.9, 0.99, 0.999\}$

Points to remember

hampath

- is used in the case of smooth functions ;
- permits to treat discontinuous optimal problems by smooth regularization ;
- computes accurately the jacobian $S'(z_0, \lambda)$;
- is easy to use since it has a MATLAB interface.

Other applications

Quantum mechanic problems

The particles are controlled by laser fields and interact with the environment.

- homotopy on physical parameters (characteristics of the particle).
- regularization by homotopy (to be smooth).

3 bodies problem

The satellite goes from the Earth to the moon.

- homotopy on the influence of the moon.
- homotopy on the maximal thrust.
- ...

Improvement

Discontinuous case

hampath can be applied with smooth functions and permits to have a good approximation of a discontinuous problem.

- Associate the code SHOOT PACKAGE [9] (shooting method for the discontinuous case) with **hampath**.
- Study the second order optimality conditions in the discontinuous case (no complete results for now).

Continuation method and bifurcations

hampath computes at each step the jacobian $S'(z_0, \lambda)$ used to find the tangent vector. We want to integrate the work of DIECI AND AL. [8] to have a path following by smooth SVD.

- Update of the decomposition of the jacobian.
- Detection of bifurcation points.

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